ON MOTIVIC PRINCIPAL VALUE INTEGRALS

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Abstract. Inspired by p-adic (and real) principal value integrals, we introduce motivic principal value integrals associated to multi-valued rational differential forms on smooth algebraic varieties. We investigate the natural question whether (for complete varieties) this notion is a birational invariant. The answer seems to be related to the dichotomy of the Minimal Model Program.

Introduction

0.1. Real and p-adic principal value integrals were first introduced by Langlands in the study of orbital integrals [Lan1] [Lan2] [LS1] [LS2]. They are associated to multi-valued differential forms on real and p-adic manifolds, respectively.

Let for instance X be a complete smooth connected algebraic variety of dimension n over $\mathbb{Q}_p$ (the field of p-adic numbers). Denoting by Ω^n_X the vector space of *rational* differential n-forms on X, take $\omega \in (\Omega^n_X)^{\otimes d}$ defined over $\mathbb{Q}_p$; we then write formally $\omega^{1/d}$ and consider it as a multi-valued rational differential form on X.

We suppose that $\operatorname{div} \omega$ is a normal crossings divisor (over $\mathbb{Q}_p$) on X; say $E_i, i \in S$, are its irreducible components. Let $\operatorname{div} \omega^{1/d} := \frac{1}{d} \operatorname{div} \omega = \sum_{i \in S}(\alpha_i - 1)E_i$, where then the $\alpha_i \in \frac{1}{d}\mathbb{Z}$. If $\omega^{1/d}$ has *no logarithmic poles*, i.e. if all $\alpha_i \neq 0$, the *principal value integral* $PV \int_{X(\mathbb{Q}_p)} |\omega^{1/d}|_p$ of $\omega^{1/d}$ on $X(\mathbb{Q}_p)$ is defined as follows. Cover $X(\mathbb{Q}_p)$ by (disjoint) small enough open balls B on which there exist local coordinates $x_1, \dots, x_n$ such that all E_i are coordinate hyperplanes. Consider for each B the *converging* integral $\int_B |x_1 x_2 \cdots x_n|_p^s |\omega^{1/d}|_p$ for $s \in \mathbb{C}$ with $\mathcal{R}(s) >> 0$, take its meromorphic continuation to $\mathbb{C}$ and evaluate this in $s = 0$; then add all these contributions. One can check that the result is independent of all choices.

In the real setting we proceed similarly but then we also need a partition of unity, and we have to assume that $\omega^{1/d}$ has no integral poles, i.e. the $\alpha_i \notin \mathbb{Z}_{\leq 0}$. Here the independency result is somewhat more complicated; it was verified in detail in [Ja1].

0.2. These principal value integrals appear as coefficients of asymptotic expansions of oscillating integrals and fibre integrals, and as residues of poles of distributions $|f|^\lambda$ or Igusa zeta functions. See [Ja1, §1] for an overview and [AVG][De2][Ig1][Ig2][Ja2][Lae] for more details.

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0.3. Last years (usual) p-adic integration and p-adic Igusa zeta functions were 'upgraded' to motivic integration and motivic zeta functions in various important papers of Denef and Loeser (after an idea of Kontsevich [Ko]). We mention the first papers [DL1] [DL2] and surveys [DL3][Lo][Ve4].

In this note we introduce similarly motivic principal value integrals. It is not totally clear what the most natural approach is; however the following should be satisfied. Returning to the p-adic setting of (0.1), we denote $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell\notin I} E_\ell)$ for $I \subset S$. So $X = \coprod_{I\subset S} E_I^\circ$. Then, if suitable conditions about good reduction mod p are satisfied, a similar proof as for Denef's formula for the p-adic Igusa zeta function [De1] yields that $PV \int_{X(\mathbb{Q}_p)} |\omega^{1/d}|_p$ is given (up to a power of p) by

$$\sum_{I\subset S} \sharp(E_I^\circ)_{\mathbb{F}_p} \prod_{i\in I} \frac{p-1}{p^{\alpha_i}-1},$$

where $\sharp(\cdot)_{\mathbb{F}_p}$ denotes the number of $\mathbb{F}_p$-rational points of the reduction mod p. Since motivic objects should specialize to the analogous p-adic objects (for almost all p), any decent definition of a motivic principal value integral $PV \int_X \omega^{1/d}$ associated to analogous X and $\omega^{1/d}$ (say over $\mathbb{C}$) should boil down to the formula

$$\sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{L-1}{L^{\alpha_i}-1}$$

(up to a power of L). Here $[\cdot]$ denotes the class of a variety in the Grothendieck ring of algebraic varieties, and $L := [\mathbb{A}^1]$, see (1.5). Note also that this is precisely the 'user-friendly formula' (in the terminology of [Cr]) for the *converging* motivic integral associated to the $\mathbb{Q}$-divisor $\operatorname{div} \omega^{1/d} = \sum_{i\in S}(\alpha_i - 1)E_i$ *if all* $\alpha_i > 0$. We will use (evaluations of) motivic zeta functions as in [Ve2] or [Ve3] to introduce this desired motivic principal value integral.

0.4. *Remark.* (1) Remembering the origin of principal value integrals, we mention that Hales introduced motivic orbital integrals, specializing to the usual p-adic orbital integrals [Ha1][Ha2].

(2) As in the p-adic case, the study of motivic principal value integrals, especially their vanishing, is related to determining the poles of motivic zeta functions, and hence of the derived Hodge and topological zeta functions. A nice result about the vanishing of real principal value integrals, and a conjecture in the p-adic case, is in [DJ].

0.5. Since a multi-valued differential form $\omega^{1/d}$ is in fact a birational notion, it is a natural question whether the motivic principal value integral is a birational invariant. In other words, if X_1 and X_2 are different complete smooth models of the birational equivalence class associated to $\omega^{1/d}$ such that $\operatorname{div} \omega^{1/d}$ is a normal crossings divisor and $\omega^{1/d}$ has no logarithmic poles on both X_1 and X_2, is then $PV \int_{X_1} \omega^{1/d} = PV \int_{X_2} \omega^{1/d}$?

This appears to be related to the dichotomy of the Minimal Model Program. We show by explicit counterexamples that the answer is in general negative when the Kodaira dimension is $-\infty$. On the other hand, when the Kodaira dimension is nonnegative, we prove birational invariance in dimension 2. In higher dimensions, we explain how the motivic principal value integrals yield a 'partial' birational invariant, assuming the Minimal Model Program. Here some subtle problems appear, which we think are interesting to investigate.

0.6. We also introduce motivic principal value integrals on a smooth variety X if $\operatorname{div}\omega^{1/d}$ is not necessarily a normal crossings divisor, facing similar problems. (For real principal value integrals this was considered by Jacobs [Ja1, §7].)

0.7. We will work over a field k of characteristic zero. When using minimal models we moreover assume k to be algebraically closed. In §1 we briefly recall the necessary birational geometry, and the for our purposes relevant motivic zeta function. In §2 we introduce motivic principal value integrals on smooth varieties. We first proceed on the level of Hodge polynomials to show that our approach with evaluations of motivic zeta functions can really be considered as the analogue of 'classical' real or p-adic principal value integrals. Then in §3 we consider the birational invariance question.

1. Birational geometry

As general references for §§1.1–1.4 we mention [KM] and [Ma].

1.1. An algebraic variety is an integral separated scheme of finite type over $\operatorname{Spec} k$, where k is a field of characteristic zero. A modification is a proper birational morphism. A log resolution of an algebraic variety is a modification $h : Y \to X$ from a smooth Y such that the exceptional locus of h is a (simple) normal crossings divisor.

Let B be a $\mathbb{Q}$-divisor on X. Then a log resolution of B is a modification $h : Y \to X$ from a smooth Y such that the exceptional locus of h is a divisor, and its union with $h^{-1}(\operatorname{supp} B)$ is a (simple) normal crossings divisor.

1.2. Moreover, let X be normal and denote $n := \dim X$. A (Weil) $\mathbb{Q}$-divisor D on X is called $\mathbb{Q}$-Cartier if some integer multiple of D is Cartier. And X is called $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier.

The variety X has a well-defined linear equivalence class K_X of canonical (Weil) divisors. Its representatives are the divisors $\operatorname{div}\eta$ of rational differential n-forms η on X. Denoting by Ω^n_X the vector space of those rational differential n-forms, we can consider more generally elements $\omega \in (\Omega^n_X)^{\otimes d}$ for any $d \in \mathbb{Z}_{>0}$, and their associated divisor $\operatorname{div}\omega$. Then we write formally $\omega^{1/d}$, considered as a multi-valued rational differential form on X, and put $\operatorname{div}\omega^{1/d} := \frac{1}{d}\operatorname{div}\omega$. Since $\operatorname{div}\omega$ represents dK_X, we can say that the $\mathbb{Q}$-divisor $\operatorname{div}\omega^{1/d}$ represents K_X.

One says that X is Gorenstein if K_X is Cartier, and $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

1.3. For a $\mathbb{Q}$-Gorenstein X, let $h : Y \to X$ be a log resolution of X, and denote by $E_i, i \in S$, the irreducible components of the exceptional locus of h. One says that X is *terminal* and *canonical* if in the expression

$$K_{Y|X} := K_Y - h^* K_X = \sum_{i\in S} a_i E_i$$

all $a_i, i \in S$, are greater than 0 and at least 0, respectively. (These notions are independent of the chosen resolution.) Such varieties can be considered 'mildly' singular; note that a smooth variety is terminal.

1.4. Let k be algebraically closed.

(i) A *minimal model* in a given birational equivalence class of nonnegative Kodaira dimension is a complete variety X_m in this class which is $\mathbb{Q}$-factorial and terminal and such that K_{X_m} is nef. This last condition means that the intersection number $K_{X_m} \cdot C \geq 0$ for all irreducible curves C on X_m.

The existence of these objects is the heart of Mori's Minimal Model Program. This is now accomplished in dimension ≤ 3 (and there is a lot of progress in dimension 4). In dimension 2 it is well known that there is a unique minimal model, which is moreover smooth, in each birational equivalence class. Also, each smooth complete surface in the class maps to the unique minimal model through a sequence of blowing-ups. In higher dimensions, two different minimal models are isomorphic in codimension one. Here each smooth complete variety in the class maps to a minimal model through a *rational map* (which is a composition of divisorial contractions and flips).

(ii) In a given birational equivalence class of general type (i.e. of maximal Kodaira dimension), a *canonical model* is a complete variety X_c in this class which is canonical and such that K_{X_c} is ample. This object is unique and there is a morphism from every minimal model in the class to it.

1.5. Here, by abuse of terminology, we allow a variety to be reducible.

(i) The Grothendieck ring $K_0(Var_k)$ of algebraic varieties over k is the free abelian group generated by the symbols $[V]$, where $[V]$ is a variety, subject to the relations $[V] = [V']$ if V is isomorphic to V', and $[V] = [V \setminus W] + [W]$ if W is closed in V. Its ring structure is given by $[V] \cdot [W] := [V \times W]$. (See [Bi] for alternative descriptions of $K_0(Var_k)$, and see [Po] for the recent proof that it is not a domain.) Usually, one abbreviates $L := [\mathbb{A}^1]$.

For the sequel we need to extend $K_0(\mathrm{Var}_k)$ with fractional powers of L and to localize. Fix $d \in \mathbb{Z}_{>0}$; we consider

$$K_0(\mathrm{Var}_k)[L^{-1/d}] := \frac{K_0(\mathrm{Var}_k)[T]}{(LT^d - 1)}$$

(where $L^{-1/d} := \bar{T}$). We then localize this ring with respect to the elements $L^{i/d} - 1$, $i \in \mathbb{Z} \setminus \{0\}$. What we really need is the subring of this localization generated by $K_0(\mathrm{Var}_k)$, L^{-1} and the elements $(L-1)/(L^{i/d} - 1), i \in \mathbb{Z} \setminus \{0\}$; we denote this subring by $\mathcal{R}_d$.

(ii) For a variety V, we denote by $h^{p,q}(H_c^i(V, \mathbb{C}))$ the rank of the (p,q)-Hodge component in the mixed Hodge structure of the ith cohomology group with compact support of V. The *Hodge polynomial* of V is

$$H(V) = H(V; u, v) := \sum_{p,q} \big(\sum_{i \geq 0} (-1)^i h^{p,q}(H_c^i(V, \mathbb{C})) \big) u^p v^q \in \mathbb{Z}[u, v].$$

Precisely by the defining relations of $K_0(\mathrm{Var}_k)$, there is a well-defined ring homomorphism $H : K_0(\mathrm{Var}_k) \to \mathbb{Z}[u, v]$, determined by $[V] \mapsto H(V)$. It induces a ring homomorphism H from $\mathcal{R}$ to the 'rational functions in u, v with fractional powers'.

1.6. In [DL1] Denef and Loeser associated a motivic zeta function to a regular function on a smooth variety. In [Ve2] and [Ve3, §2] we considered several generalizations; we mention here a special case of [Ve3, (2.2)].

(i) Let X be a canonical variety and D any $\mathbb{Q}$-divisor on X. Take a log resolution $h : Y \to X$ of D and denote by $E_i, i \in S$, the irreducible components of the union of $h^{-1}(\operatorname{supp} D)$ and the exceptional locus of h. For each $i \in S$ let $\nu_i - 1$ and N_i denote the multiplicity of E_i in $K_{Y|X}$ and h^*D, respectively. Note that all $\nu_i \geq 1$ since X is canonical. We also put $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$. We associated to D on X the zeta function

$$\mathcal{Z}_X(D; s) := L^{-n} \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{L-1}{L^{\nu_i + sN_i} - 1}.$$

Here L^{-s} is just the traditional notation for a variable T. So $\mathcal{Z}_X(D; s)$ lives, for example, in a polynomial ring 'with fractional powers' in a variable T over some ring $\mathcal{R}_d$, localized with respect to the elements $L^\nu T^N - 1$ and $L^\nu - T^N$ for $\nu \in \frac{1}{d}\mathbb{Z}$ (and $\nu \geq 1$) and $N \in \mathbb{Q}_{>0}$. (We verified that the defining expression does not depend on the chosen resolution using the weak factorization theorem [AKMW][Wł].)

(ii) One can specialize $\mathcal{Z}_X(D; s)$ to the Hodge polynomial level via the map H, obtaining

$$Z_X(D; s) = (uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1},$$

where now $(uv)^{-s}$ is a variable.

(iii) For any constructible subset W of X, we can consider more generally zeta functions $\mathcal{Z}_{W\subset X}(D; s)$ and $Z_{W \subset X}(D; s)$, using $E_I^\circ \cap h^{-1}W$ instead of E_I° in the defining expressions.

Note. Here we re-normalized the zeta functions of [Ve3, §2] with a factor L^{-n} and $(uv)^{-n}$, respectively.

2. Motivic principal value integrals on smooth varieties

2.1. Let Y be a smooth algebraic variety of dimension n. Let $\omega^{1/d}$ be a multi-valued differential form on Y, such that $\operatorname{div} \omega^{1/d}$ is a normal crossings divisor and $\omega^{1/d}$ has no logarithmic poles.

Denote by $E_i, i \in S$, the irreducible components of $E = \operatorname{supp}(\operatorname{div} \omega^{1/d})$, and let $\alpha_i - 1$ be the multiplicity of E_i in $\operatorname{div}(\omega^{1/d})$. So $\operatorname{div} \omega^{1/d} = \sum_{i\in S} (\alpha_i - 1) E_i$, and the $\alpha_i \in \frac{1}{d}\mathbb{Z} \setminus \{0\}$. For $I \subset S$ we put $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$. Note that $Y = \coprod_{I \subset S} E_I^\circ$.

2.2. We start by giving two equivalent definitions for the principal value integral of $\omega^{1/d}$ on Y on the level of Hodge polynomials. The first one is analogous to the classical real and p-adic situation, and the second one will turn out to be a specialization of the definition on the motivic level.

(1) Consider for $s \in \mathbb{Z}, s >> 0$, the 'motivic integral on Hodge polynomial level'

$$I(s) := \int_{\mathcal{L}(Y)} (uv)^{-\operatorname{ord}_t(\operatorname{div} \omega^{1/d} + sE)} d\mu,$$

where $\mathcal{L}(Y)$ is the arc space of Y and $\mathrm{ord}_t(\cdot)$ denotes the order of the given divisor along an arc in $\mathcal{L}(Y)$; see e.g. [DL2][Ba][Ve2]. This is the analogue of the converging integral for $s >> 0$ in the classical case. By [Ba] or [DL2] also $I(s)$ converges for $s >> 0$ (in fact if and only if $\alpha_i + s > 0$ for all $i \in S$), and then

$$I(s) = (uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{(uv-1)(uv)^{-s}}{(uv)^{\alpha_i} - (uv)^{-s}}.$$

Consider now the unique rational function $Z(T)$ over $\mathbb{Q}(u^{1/d}, v^{1/d})$ in the variable T, yielding $I(s)$ when evaluated in $T = (uv)^{-s}$ for all $s >> 0$; it is given by

$$Z(T) = (uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{(uv-1)T}{(uv)^{\alpha_i} - T}.$$

Hodge level definition 1. The principal value integral of $\omega^{1/d}$ on Y is $\lim_{T \to 1} Z(T) = ev_{T=1} Z(T)$ and is thus given by the formula

$$(uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{uv-1}{(uv)^{\alpha_i} - 1}.$$

Note that this proces is indeed analogous to the classical case, where we take the limit for $s \to 0$ of a meromorphic continuation.

(2) We consider the zeta function $Z_Y(\operatorname{div} \omega^{1/d}; s)$ of (1.6(ii)). Since here $\operatorname{div} \omega^{1/d}$ is already a normal crossings divisor on the smooth variety Y, we have

$$Z_Y(\operatorname{div} \omega^{1/d}; s) = (uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{uv-1}{(uv)^{1+(\alpha_i - 1)s} - 1}.$$

(If you don't like fractional powers of $T = (uv)^{-s}$, just consider $T^{1/d}$ as a variable with integer powers $-d(\alpha_i - 1)$.)

Hodge level definition 2. The principal value integral of $\omega^{1/d}$ on Y is

$$\lim_{s \to 1} Z_Y(\operatorname{div} \omega^{1/d}; s) = ev_{s=1} Z_Y(\operatorname{div} \omega^{1/d}; s).$$

This means of course evaluating in $T = (uv)^{-1}$, and yields the same formula as in the previous definition.

Remark. Alternatively, we could have taken the (re-normalized) zeta function of [Ve2], associated to the effective divisor aE and the sheaf of *regular* differential forms $\mathcal{O}(aE) \otimes \omega^{1/d}$ for some $a >> 0$, which is given by the formula

$$(uv)^{-n} \sum_{I \subset S} H(E_I^\circ) \prod_{i \in I} \frac{uv-1}{(uv)^{a+\alpha_i+sa} - 1},$$

and evaluate it in $s=-1$.

2.3. On the level of the Grothendieck ring, we cannot use the first approach since zero divisors may occur. The second approach however generalizes and yields the desired formula.

Definition. The motivic principal value integral of $\omega^{1/d}$ on Y is the evaluation of $\mathcal{Z}_Y(\operatorname{div}\omega^{1/d};s)$ in $s=1$; it is given by the formula

$$L^{-n}\sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{L-1}{L^{\alpha_i}-1},$$

living in the ring $\mathcal{R}_d$ of (1.5). We denote it by $PV\int_Y\omega^{1/d}$.

Remarks. (1) One easily verifies that evaluating $\mathcal{Z}_Y(\operatorname{div}\omega^{1/d};s)$ in $s=1$ (i.e. in $T=L^{-1}$) indeed yields a well defined element in $\mathcal{R}_d$.

(2) In the special case that all $\alpha_i>-1$, we can just use the converging motivic integral $\int_{\mathcal{L}(Y)}L^{-\operatorname{ord}_t(\operatorname{div}\omega^{1/d})}d\mu$, given by the same formula, but then it is only well defined in a completion of $K_0(\mathrm{Var}_k)[L^{-1/d}]$, see [DL2] and [Ve2].

(3) For a constructible subset W of Y we can consider more generally $PV\int_{W\subset Y}\omega^{1/d}$ as the evaluation of $\mathcal{Z}_{W\subset Y}(\operatorname{div}\omega^{1/d};s)$ in $s=1$, see (1.6(iii)). This corresponds morally to 'classical' p-adic and real principal value integrals involving a locally constant function and C^∞ function, respectively, with compact support.

2.4. Let now X be a smooth algebraic variety of dimension n, and $\omega^{1/d}$ any multi-valued differential form on X. Is it possible to associate a well defined 'natural' principal value integral to $\omega^{1/d}$? Of course in some sense logarithmic poles will have to be excluded.

A first natural idea is to consider (the pull-back of) $\omega^{1/d}$ via a modification $h:Y\to X$ such that the divisor of $h^*\omega^{1/d}$ is a normal crossings divisor on Y. If there exists a modification for which $h^*\omega^{1/d}$ has no logarithmic poles on Y, then the desired principal value integral could be defined as $PV\int_Y h^*\omega^{1/d}$. Of course the point here is whether this is independent of the chosen modification.

Looking at the zeta function approach in Definition 2.3, another natural idea is just to define the principal value integral of $\omega^{1/d}$ on X as the evaluation of $\mathcal{Z}_X(\operatorname{div}\omega^{1/d};s)$ in $s=1$, *if this makes sense.* Here no choices are involved.

We verify that the first approach works and yields the same result as the second approach *if* there exists a modification $h:Y\to X$ satisfying a slightly stronger condition than above. Then we indicate some subtle problems concerning the 'naive' first approach.

2.5. We say that a modification $h:Y\to X$ is *good* if it is a log resolution of $D:=\operatorname{div}\omega^{1/d}$ on X, for which $\operatorname{div}(h^*\omega^{1/d})$ has no logarithmic poles on Y.

Note that there could exist modifications $h:Y\to X$ for which $\operatorname{div}(h^*\omega^{1/d})$ is a normal crossings divisor and $h^*\omega^{1/d}$ has no logarithmic poles, but such that h^*D is *not* a normal crossings divisor ! Indeed, it is possible that $h^*\omega^{1/d}$ has multiplicity zero along some component of h^*D, meaning that this component does not occur in $\operatorname{div}(h^*\omega^{1/d})$. For example, let D (locally) be given by $\frac{1}{2}D_1 - \frac{3}{2}D_2$, where D_1 and D_2 are smooth curves, intersecting each other in a point P with intersection multiplicity 2, see Figure 1. Let h be the blowing-up of X with centre P. Then $h^*\omega^{1/d}$ has multiplicity zero along the exceptional curve D_3 of h; so $\operatorname{div}(h^*\omega^{1/d})$ is (locally) a normal crossings divisor but h^*D is not.

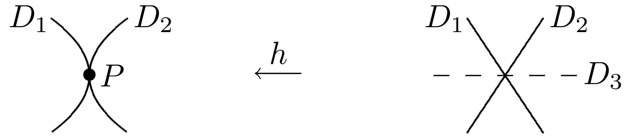

Figure 1

2.6. Proposition. *Suppose that there exists a good modification $h : Y \to X$. Denote by $E_i, i \in S$, the irreducible components of $h^{-1}D$, and let $\alpha_i - 1$ be the multiplicity of E_i in $\operatorname{div}(h^*\omega^{1/d})$. Put $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$. Then the evaluation of $\mathcal{Z}_X(\operatorname{div}\omega^{1/d}; s)$ in $s = 1$ is well defined, it is equal to $PV \int_Y h^*\omega^{1/d}$, and is given by the formula $L^{-n}\sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{L-1}{L^{\alpha_i}-1}$.*

Note. Here $\alpha_i = 1$ could occur, meaning that E_i does not appear in $\operatorname{div}(h^*\omega^{1/d})$.

Proof. Denote the multiplicities of E_i in $K_{Y|X}$ and in h^*D by $\nu_i - 1$ and N_i, respectively. Since h is really a log resolution of D, we can express $\mathcal{Z}_X(D; s)$ as

$$L^{-n}\sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{L-1}{L^{\nu_i+sN_i}-1}.$$

Now $D = \operatorname{div}\omega^{1/d}$ and $\operatorname{div}(h^*\omega^{1/d})$ are representatives of K_X and K_Y, respectively. Hence $\operatorname{div}(h^*\omega^{1/d}) = K_{Y|X} + h^*D$, meaning that $\alpha_i = \nu_i + N_i$ for all $i \in S$. Since $\operatorname{div}(h^*\omega^{1/d})$ has no logarithmic poles, all $\alpha_i \neq 0$. So indeed evaluating $\mathcal{Z}_X(D; s)$ in $s = 1$ makes sense and yields the stated formula, which is just $PV \int_Y h^*\omega^{1/d}$. □

We define the principal value integral of $\omega^{1/d}$ on X as given by Proposition 2.6. For completeness we recall all data.

2.7. Definition. Let X be a smooth algebraic variety of dimension n and $\omega^{1/d}$ a multi-valued differential form on X. Suppose that there exists a log resolution $h : Y \to X$ of $\operatorname{div}\omega^{1/d}$ on X, for which $h^*\omega^{1/d}$ has no logarithmic poles on Y. Then the principal value integral of $\omega^{1/d}$ on X, denoted $PV \int_X \omega^{1/d}$, is given by one of the equivalent expressions in Proposition 2.6.

Remarks. (1) Also here we could proceed alternatively using the zeta functions of [Ve2].

(2) We can proceed more generally, involving a constructible subset W of X, just as in Remark (3) after Definition 2.3.

(3) For real principal value integrals, Jacobs gave a similar definition [Ja1, §7].

2.8. We return to the first approach. Suppose now that there exists a modification $g : Z \to X$ such that $\operatorname{div}(g^*\omega^{1/d})$ is a normal crossings divisor and $g^*\omega^{1/d}$ has no logarithmic poles on Z. (Recall that then g is not necessarily good.) Two subtle questions impose themselves here.

I) Suppose that there exists at least one good modification of X; so $PV\int_X \omega^{1/d}$ is defined. Is this principle value integral then equal to the obvious formula associated to $g^*\omega^{1/d}$ on Z ? I.e., with $\operatorname{div}(g^*\omega^{1/d}) = \sum_{i\in S_Z}(\alpha_i - 1)E_i$ and E_I° for $I \subset S_Z$ as usual, is

$$PV\int_X \omega^{1/d} = L^{-n}\sum_{I\subset S_Z}[E_I^\circ]\prod_{i\in I}\frac{L-1}{L^{\alpha_i}-1}\ ?$$

If there exists a modification $\pi : Y \to Z$ such that $h := g\circ\pi : Y \to X$ is good, then the answer is yes. This can be verified using the zeta function $\mathcal{Z}_Z(\operatorname{div}(g^*\omega^{1/d}); s)$, for which we use the defining expressions on both Z and Y. One easily computes that evaluating in $s = 1$ yields the right and left hand sides above, respectively. The point is that 'deleting irreducible components of $h^*(\operatorname{div}\omega^{1/d})$ on Y with $\alpha = 1$' does not change the formula for $PV\int_X \omega^{1/d}$ in terms of h.

When there does not exist such a modification $Y \to Z$, we do not know the answer.

II) Suppose on the other hand that *no* good modification of X exists. Are the expressions

$$L^{-n}\sum_{I\subset S_Z}[E_I^\circ]\prod_{i\in I}\frac{L-1}{L^{\alpha_i}-1}$$

for modifications $Z \to X$ as above *independent* of the chosen Z ?

2.9. The principal value integrals of Definition 2.3 and the more general Definition 2.7 satisfy a 'Poincaré duality'. Bittner [Bi] showed that there exists a ring involution $\mathcal{D}$ of $K_0(\mathrm{Var}_k)[L^{-1}]$ satisfying $\mathcal{D}(L) = L^{-1}$, and characterized by $\mathcal{D}([X]) = L^{-\dim X}[X]$ when X is a complete connected smooth variety. (It is in fact a lifting of the usual duality operator on the level of motives to the level of varieties. And specializing $\mathcal{D}$ to the level of Hodge polynomials is just a reformulation of Poincaré and Serre duality.) It extends to the rings $\mathcal{R}_d$ via $\mathcal{D}(L^{1/d}) = L^{-1/d}$.

Proposition. *The principal value integrals of Definitions 2.3 and 2.7 satisfy*

$$\mathcal{D}(PV\int_X \omega^{1/d}) = L^{-\dim X} PV\int_X \omega^{1/d}.$$

Proof. This follows from the concrete formula for $PV\int_X \omega^{1/d}$ by the same computation as in e.g. [Ba], [DM] or [Ve3]. □

3. Birational invariance ?

Here we assume k to be algebraically closed.

3.1. Actually, a (multi-valued) differential form is a birational notion. When we consider such a form $\omega^{1/d}$ on a variety X and its pull-back $h^*\omega^{1/d}$ on Y via a modification $h : Y \to X$, this is just a matter of notation : $\omega^{1/d}$ and $h^*\omega^{1/d}$ are in fact the same element in a one-dimensional vector space over the function field of X.

Fix a form $\omega^{1/d}$ for which there exists a smooth complete variety X such that $\operatorname{div}(\omega^{1/d})$ on X is a normal crossings divisor and $\omega^{1/d}$ has no logarithmic poles on X. Then we can consider $PV\int_X \omega^{1/d}$, and it is a natural question whether this notion depends on the chosen such model X. In other words : is the motivic principal value integral a birational invariant ?

3.2. *Remark.* A necessary condition is of course that, if $\pi : X' \to X$ is the blowing-up of an X as above in a smooth centre that has normal crossings with $\operatorname{div}(\omega^{1/d})$ and such that $\omega^{1/d}$ has also no logarithmic poles on X', then $PV\int_X \omega^{1/d} = PV\int_{X'} \omega^{1/d}$. This can be verified by straightforward computations as in [Ve1], [Ve3, Lemma 2.3.2] or [Al].

We should remark that this is however not sufficient to derive birational invariance with the help of the weak factorization theorem. Indeed, on some 'intermediate' varieties connecting two such models the form $\omega^{1/d}$ could have logarithmic poles.

3.3. Note that in dimension one there is only one smooth complete model in a given birational equivalence class. So from now on we work in dimension at least two.

First we show that when the Kodaira dimension is $-\infty$, the answer is in general negative.

3.4. *Example.* We work in the class of rational surfaces and take $\omega^{1/2}$ on $\mathbb{P}^2_{(X:Y:Z)}$, given by $\omega^{1/2} = y^{-3/2}dxdy$ on the affine chart $\mathbb{A}^2_{(x,y)}$. Then on the chart $\mathbb{A}^2_{(y,z)}$ we have that $\omega^{1/2} = y^{-3/2}z^{-3/2}dydz$. Denoting $C_1 := \{Y = 0\}$ and $C_2 := \{Z = 0\}$, we see that $\operatorname{div}(\omega^{1/2}) = -\frac{3}{2}C_1 - \frac{3}{2}C_2$ is a normal crossings divisor on $\mathbb{P}^2$ and that no logarithmic poles occur.

Consider now the birational map $\pi : \mathbb{P}^2_{(X':Y':Z')} - \to \mathbb{P}^2_{(X:Y:Z)}$ given by $\mathbb{A}^2_{(x',y')} \to \mathbb{A}^2_{(x,y)} : (x',y') \mapsto (x', y'-x'^2)$. On $\mathbb{P}^2_{(X':Y':Z')}$ our form $\omega^{1/2}$ is given by $(y'-x'^2)^{-3/2}dx'dy'$ and $(y'z'-1)^{-3/2}dy'dz'$ on the analogous charts. Hence on the 'new' $\mathbb{P}^2$ we have that $\operatorname{div}(\omega^{1/2}) = -\frac{3}{2}C_1$, where we denote the birational transform of C_1, i.e. $\{Y'Z' - X'^2 = 0\}$, by the same symbol. The formula of Definition 2.3 yields

$$PV\int_{\mathbb{P}^2_{(X:Y:Z)}} \omega^{1/2} = L^{-2}\big(L^2 - L + 2L\frac{L-1}{L^{-1/2}-1} + \frac{(L-1)^2}{(L^{-1/2}-1)^2}\big) = 0$$

and

$$PV\int_{\mathbb{P}^2_{(X':Y':Z')}} \omega^{1/2} = L^{-2}\big(L^2 + (L+1)\frac{L-1}{L^{-1/2}-1}\big) = -L^{-3/2}(L + L^{1/2} + 1) \neq 0.$$

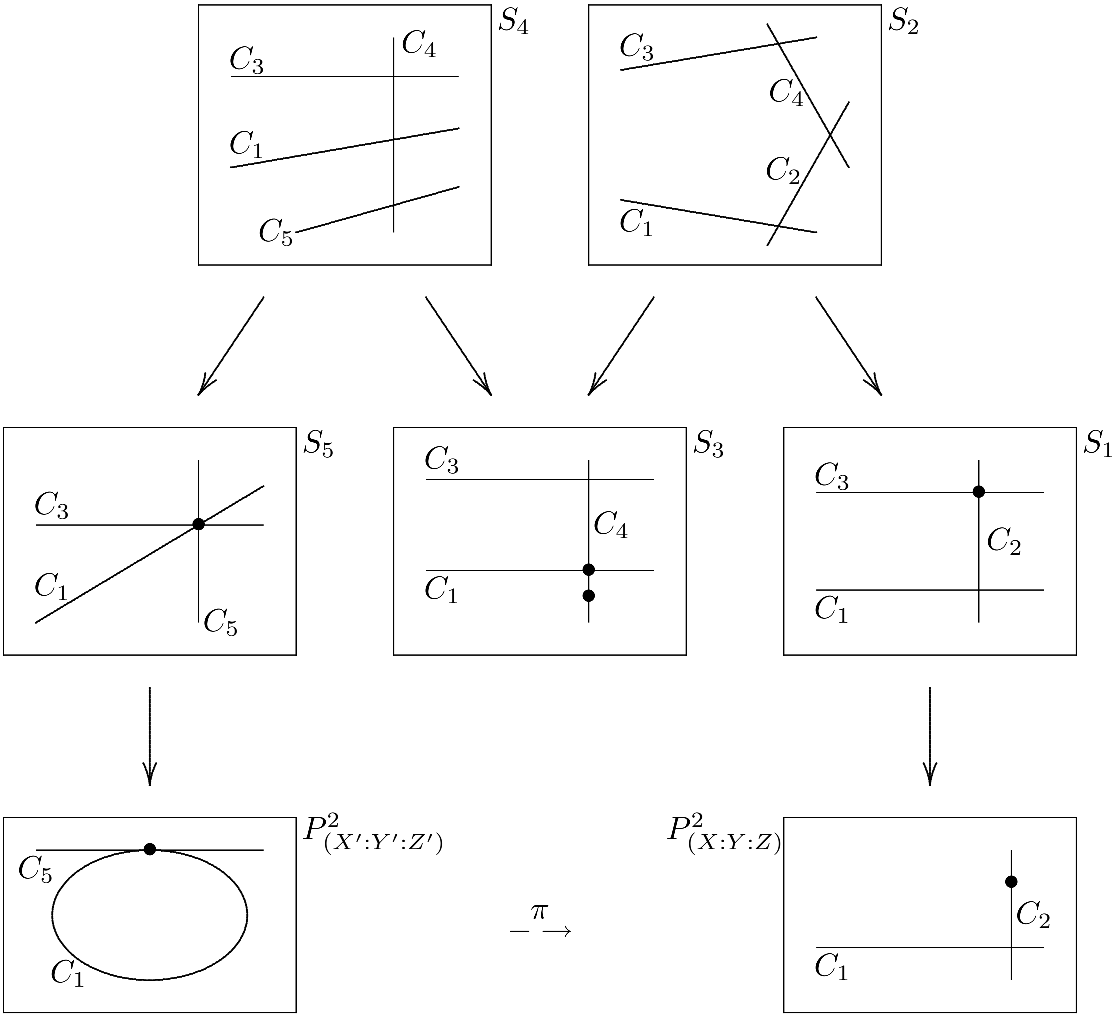

Figure 2

It is useful to indicate the 'geometric reason' why this happens. We decompose in Figure 2 the map π in a composition of blowing-ups and blowing-downs, where the fat points indicate the centers of blowing-up and C_3, C_4 and C_5 are exceptional curves. The surfaces on the middle row are ruled surfaces. One easily verifies that the multiplicities of C_3, C_4 and C_5 in $\operatorname{div} \omega^{1/2}$ are $-\frac{1}{2}, -1$ and 0, respectively. This means in particular that C_5 does not occur in the support of $\operatorname{div} \omega^{1/2}$ on S_4, S_5 and $\mathbb{P}^2_{(X':Y':Z')}$ (where C_5 is a line), and that $\operatorname{div} \omega^{1/2}$ has logarithmic poles on S_2, S_3 and S_4. So we *cannot* identify $PV \int_{\mathbb{P}^2_{(X:Y:Z)}} \omega^{1/2} = PV \int_{S_1} \omega^{1/2}$ with $PV \int_{\mathbb{P}^2_{(X':Y':Z')}} \omega^{1/2} = PV \int_{S_5} \omega^{1/2}$ via principal value integrals on the intermediate surfaces S_2, S_3 and S_4 since they are *not defined* there.

Note. (i) We found this example several years ago; it was briefly mentioned by Jacobs [Ja1, §8] in the context of real principal value integrals.

(ii) Actually, it is also valid when k is not algebraically closed. (And if we would have introduced principal value integrals in arbitrary characteristic by the same formula, it would still work.)

3.5. Example 3.4 can be adapted to the birational equivalence class of any non-rational ruled surface; then we use only the middle and top row of Figure 2. It is possible to give a similar form $\omega^{1/2}$ with $PV \int_{S_1} \omega^{1/2} = 0$ and $PV \int_{S_5} \omega^{1/2} \neq 0$ (maybe $\mathrm{div}(\omega^{1/2})$ will contain more fibres in its support). So for surfaces such examples exist in every birational equivalence class of Kodaira dimension $-\infty$.

Moreover, by taking Cartesian products with arbitrary complete smooth varieties, Example 3.4 can be extended to arbitrary dimension.

We now turn to the other case, i.e. when the Kodaira dimension is nonnegative.

3.6. Theorem. *Fix the birational equivalence class of a surface of nonnegative Kodaira dimension, and a multivalued differential form $\omega^{1/d}$ on it. Suppose that this class contains a smooth complete model X such that $\mathrm{div}(\omega^{1/d})$ is a normal crossings divisor on X and $\omega^{1/d}$ has no logarithmic poles on X. Then $PV \int_X \omega^{1/d}$ is independent of the chosen such model, and is thus a birational invariant.*

Proof. (Recall that for smooth surfaces complete is equivalent to projective.) Let X_m be the unique (smooth, projective) minimal model in the class, and let $h : Y \to X_m$ be the composition of the minimal set of blowing-ups, needed to make $\mathrm{div}\, \omega^{1/d}$ a normal crossings divisor on Y. More precisely, if $\mathrm{div}\, \omega^{1/d}$ is a normal crossings divisor on X_m, put $Y := X_m$. Otherwise, let $h_1 : Y_1 \to X_m$ be obtained from X_m by blowing up the finite number of points where $\mathrm{div}\, \omega^{1/d}$ has no normal crossings. If $\mathrm{div}\, \omega^{1/d}$ is a normal crossings divisor on Y_1, put $Y := Y_1$. Otherwise, continuing this way abuts in the unique smooth projective Y such that $\mathrm{div}\, \omega^{1/d}$ is a normal crossings divisor on Y and $h : Y \to X_m$ is minimal with respect to this property.

Suppose now that X is any model as in the énoncé of the theorem. Since $\mathrm{div}\, \omega^{1/d}$ is a normal crossings divisor on X, there is a morphism $\pi : X \to Y$. Also, since $\omega^{1/d}$ has no logarithmic poles on X, the same is certainly true on Y. Hence also $PV \int_Y \omega^{1/d}$ is defined, and is equal to $PV \int_X \omega^{1/d}$ by Remark 3.2. □

3.7. In higher dimensions we face the non-existence of a unique minimal model, and the fact that in general a (smooth, complete) variety does not map to a minimal model by a *morphism.* A reasonable idea is to try to adapt Definition 2.7 as follows, assuming the Minimal Model Program. Take a birational equivalence class of nonnegative Kodaira dimension and a multi-valued differential form $\omega^{1/d}$ on it.

Suppose that there exists a minimal model X in the given class, and a log resolution $h : Y \to X$ of $\mathrm{div}\, \omega^{1/d}$ on X, for which $\omega^{1/d}$ has no logarithmic poles on Y. Define then the (candidate) birational invariant associated to $\omega^{1/d}$ as $PV \int_Y \omega^{1/d}$. Now independence of both X and Y has to be checked.

Fixing such a minimal model X, the independence of Y is proven by the same argument as for Proposition 2.6. Indeed, $PV \int_Y \omega^{1/d}$ is just the evaluation of $\mathcal{Z}_X(\mathrm{div}\, \omega^{1/d}; s)$ in $s = 1$. (Recall that this zeta function was defined more generally on canonical varieties, so certainly on minimal models.) We now verify that the zeta function $\mathcal{Z}_X(\mathrm{div}\, \omega^{1/d}; s)$ itself in fact does not depend on the chosen X; then a fortiori the same is true for its evaluation in $s = 1$.

3.8. Proposition. *Let $\omega^{1/d}$ be a multi-valued differential form on a birational equivalence class of nonnegative Kodaira dimension. Let X_1 and X_2 be minimal models in this class; then $\mathcal{Z}_{X_1}(\operatorname{div}\omega^{1/d}; s) = \mathcal{Z}_{X_2}(\operatorname{div}\omega^{1/d}; s)$.*

Proof. Since X_1 and X_2 are isomorphic in codimension one, $\operatorname{div}\omega^{1/d}$ on X_1 and X_2 are each others birational transform. Take a common log resolution $h_i : Y \to X_i$ of $\operatorname{div}\omega^{1/d}$ on X_1 and X_2. Remembering that these two ($\mathbb{Q}$-)divisors are representatives of K_{X_1} and K_{X_2}, we have $h_1^*(\operatorname{div}\omega^{1/d}) = h_2^*(\operatorname{div}\omega^{1/d})$ and $K_{Y|X_1} = K_{Y|X_2}$, see [KM, Proof of Theorem 3.52] or [Wa, Corollary 1.10].

So, computing both zeta functions by their defining expressions on Y, see (1.6), yields exactly the same formula. □

Note. When our birational equivalence class is of general type, we can use its *unique* canonical model X_c and work with $\mathcal{Z}_{X_c}(\operatorname{div}\omega^{1/d}; s)$, which shortens the argument.

Summarizing, we obtained the following well defined invariant.

3.9. Definition. Let $\omega^{1/d}$ be a multi-valued differential form on a birational equivalence class of nonnegative Kodaira dimension. Assume the Minimal Model Program. Suppose that there exists a minimal model X and a log resolution $h : Y \to X$ of $\operatorname{div}\omega^{1/d}$ on X, for which $\omega^{1/d}$ has no logarithmic poles on Y. Then $PV\int_Y \omega^{1/d}$ is independent of all choices and is thus a (partial) birational invariant of $\omega^{1/d}$.

3.10. We mention 'partial' because we are confronted with similar subtle problems as in §2. Suppose that there exists a smooth complete model Z in the given class, for which $\operatorname{div}\omega^{1/d}$ is a normal crossings divisor on Z and $\omega^{1/d}$ has no logarithmic poles on Z.

I) If there exists a minimal model X and a log resolution $Y \to X$ as in Definition 3.9, is the invariant above then equal to $PV\int_Z \omega^{1/d}$?

II) If on the other hand no such X and Y exist, are the expressions $PV\int_Z \omega^{1/d}$ the same for different such models Z ?

3.11. *Remark.* An alternative point of view for 'birational invariance' is 'independence of chosen completion' for principal value integrals on non-complete smooth varieties. For real principal value integrals Jacobs [Ja1, §8] mentioned Example 3.4 in this context.

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