POLES OF ARCHIMEDEAN ZETA FUNCTIONS FOR ANALYTIC MAPPINGS

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Abstract. Let $\boldsymbol{f}=(f_1,\ldots,f_l):U\rightarrow K^l$, with $K=\mathbb{R}$ or $\mathbb{C}$, be a K-analytic mapping defined on an open set $U\subseteq K^n$, and let Φ be a smooth function on U with compact support. In this paper, we give a description of the possible poles of the local zeta function attached to $(\boldsymbol{f},\Phi)$ in terms of a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}=(f_1,\ldots,f_l)$. When $\boldsymbol{f}$ is a non-degenerate mapping, we give an explicit list for the possible poles of $Z_\Phi(s,\boldsymbol{f})$ in terms of the equations of the supporting hyperplanes of a Newton polyhedron attached to $\boldsymbol{f}$. These results extend the corresponding results of Varchenko to the case $l\geq 1$ and $K=\mathbb{R}$ or $\mathbb{C}$.

1. Introduction

We take $K=\mathbb{R}$ or $\mathbb{C}$. Let $\boldsymbol{f}=(f_1,\ldots,f_l):U\rightarrow K^l$ be a K-analytic mapping defined on an open U in K^n. Let $\Phi:U\rightarrow\mathbb{C}$ be a smooth function on U with compact support. Then the local zeta function attached to $(\boldsymbol{f},\Phi)$ is defined as

$$Z_\Phi(s,\boldsymbol{f})=\int_{K^n\smallsetminus\boldsymbol{f}^{-1}(0)}\Phi(x)\,|\boldsymbol{f}(x)|_K^s\,|dx|,$$

for $s\in\mathbb{C}$ with $\operatorname{Re}(s)>0$, where $|dx|$ is the Haar measure on K^n. The local zeta functions have a meromorphic continuation to the whole complex plane. In the case $l=1$, the meromorphic continuation of $Z_\Phi(s,\boldsymbol{f})$ was established jointly by Bernstein and Gel'fand [3], independently by Atiyah [2], then by a different method by Bernstein [4]. In [10], see also [11], Igusa developed a uniform theory for local zeta functions over local fields of characteristic zero. In this context, there exist asymptotic expansions for oscillatory integrals depending on one parameter which are controlled by the poles of 'twisted versions' of $Z_\Phi(s,\boldsymbol{f})$, see also [1], [20]. In [18] , with $l\geq 1$ and $K=\mathbb{C}$, Phong and Sturm studied the stability of the poles of $Z_\Phi(s,\boldsymbol{f})$, under small perturbations of $\boldsymbol{f}$.

In this paper, we give a geometric description of the possible poles of $Z_\Phi(s,\boldsymbol{f})$, including the largest one, in terms of a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}}=(f_1,\ldots,f_l)$, see Theorem 2 and Proposition 2. To the best of our knowlege a such description has not been reported before, in the Archimedean context. (See however Remark 1 and Proposition 1 when $K=\mathbb{R}$.) When $\boldsymbol{f}$ is a non-degenerate mapping in the sense of [21], we give an explicit list for the possible poles of $Z_\Phi(s,\boldsymbol{f})$ in terms of the equations of the supporting hyperplanes of a Newton polyhedron

1991 *Mathematics Subject Classification.* Primary: 11M41, Secondary: 32S05, 14B05, 14M25.

Key words and phrases. Local zeta functions, analytic mappings, log canonical threshold, log-principalization of ideals, Newton polyhedra, toric varieties.

attached to $\boldsymbol{f}$, see Theorem 3. These results extend the corresponding results of Varchenko in [20] for $l=1$ and $K=\mathbb{R}$ to the case $l\geq 1$, and $K=\mathbb{R}$ or $\mathbb{C}$.

At this point we must mention that in the case $K=\mathbb{R}$ a description of the poles of $Z_\Phi(s,\boldsymbol{f})$ can be obtained by using an embedded resolution of singularities of $\sum_{i=1}^{l} f_i^2$, see [1], [11], [20]. In particular, one could define $\boldsymbol{f}$ to be non-degenerate as meaning $\sum_{i=1}^{l} f_i^2$ to be non-degenerate in the usual sense, and thus one could use all the results of [20]. We note however that there exist many mappings $\boldsymbol{f}$ which are non-degenerate in the sense of our Definition 1 but such that $\sum_{i=1}^{l} f_i^2$ are degenerate in the usual sense, see Remark 4. Thus, our approach gives a finer explicit description of the poles of $Z_\Phi(s,\boldsymbol{f})$. By combining the results presented here and those of [21], one obtains a geometric description of the poles of local zeta functions attached to analytic mappings, in Archimedean and non-Archimedean settings, in terms of log-principalization of ideals.

2. Local Zeta Functions for Analytic Mappings

2.1. **Fixing the data.** We take $K=\mathbb{R}$ or $\mathbb{C}$. For $a=(a_1,\ldots,a_n)\in K^n$ we put $|a|_K=|a|_{\mathbb{R}}$ or $|a|_{\mathbb{C}}^2$, where $|\cdot|_{\mathbb{R}}$ and $|\cdot|_{\mathbb{C}}$ are the standard norms of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

Let $f_1,\ldots,f_l$ be polynomials in $K[x_1,\ldots,x_n]$, or, more generally, $K-$analytic functions on an open set $U\subset K^n$. We consider the mapping $\boldsymbol{f}=(f_1,\ldots,f_l): K^n\to K^l$, respectively, $U\to K^l$. Let $\Phi: K^n\to\mathbb{C}$ be a smooth function with compact support, i.e. $\Phi\in C_0^\infty$, with support in U in the second case.

2.2. **Log-principalization of ideals.** We state the version of log-principalization of ideals that we will use in this paper, [7], see also [9], [22].

Theorem 1 ([7])**.** *Let $K=\mathbb{R}$ or $\mathbb{C}$ and let U be an open submanifold of K^n. Let $f_1,\ldots,f_l$ be $K-$analytic functions on U such that the ideal $\mathcal{I}_{\boldsymbol{f}}=(f_1,\ldots,f_l)$ is not trivial. Then there exists a log-principalization $h: X_K\to U$ of $\mathcal{I}_{\boldsymbol{f}}$, that is,*
(1) X_K is an $n-$dimensional $K-$analytic manifold, h is a proper $K-$analytic map which can be chosen as a composition of a finite number of blow-ups in closed submanifolds, and which is an isomorphism outside of the common zero set S_K of $f_1,\ldots,f_l$;
(2) $h^{-1}(S_K)=\cup_{i\in T}E_i$, where the E_i are closed submanifolds of X_K of codimension one, each equipped with a pair of positive integers (N_i,v_i) satisfying the following. At every point b of X_K there exist local coordinates $(y_1,\ldots,y_n)$ on X_K around b such that, if $E_1,\ldots,E_r$ are the E_i containing b, we have on some neighborhood of b that E_i is given by $y_i=0$ for $i=1,\ldots,r$,

$$h^*(\mathcal{I}_{\boldsymbol{f}})\ \textit{is generated by}\ \varepsilon(y)\prod_{i=1}^{r} y_i^{N_i},$$

and

$$h^*(dx_1\wedge\ldots\wedge dx_n)=\eta(y)\left(\prod_{i=1}^{r} y_i^{v_i-1}\right)dy_1\wedge\ldots\wedge dy_n,$$

where $\varepsilon(y)$, $\eta(y)$ are units in the local ring of X_K at b.

The (N_i,v_i), $i\in T$, are called *the numerical data of h for $\mathcal{I}_{\boldsymbol{f}}$.*

2.3. **Poles of local zeta functions.** From now on we suppose that $\boldsymbol{f}^{-1}(0) \neq \emptyset$. To $\boldsymbol{f}$ and Φ as in 2.1 we associate the local zeta function $Z_\Phi(s, \boldsymbol{f})$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, defined in the introduction.

Remark 1. (1) When $K = \mathbb{R}$ the zeta function $Z_\Phi(s, f)$ of the mapping f is clearly equal to the zeta function $Z_\Phi(\frac{s}{2}, F)$ of the function $F := \sum_{i=1}^{l} f_i^2$. In particular, it is known that $Z_\Phi(s, f)$ has a meromorphic continuation to the whole complex plane, and that its poles are negative rational numbers and of order at most n, see e.g. [13]. Also the list of candidate poles in Theorem 2 below for $K = \mathbb{R}$ can in fact be derived from the function case, see Proposition 1. But since the proof of Theorem 2 for $K = \mathbb{C}$ is also valid for $K = \mathbb{R}$ we prefer to state and prove it simultaneously for both fields.

(2) Over $K = \mathbb{C}$ the meromorphic continuation can be analogously reduced to the case of one real–analytic function. The point of Theorem 2 is the description of the candidate poles in terms of a principalization of the ideal.

Theorem 2. *Let $\boldsymbol{f}$ and Φ be as in 2.1. Let $h : X_K \to U$ be a fixed log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$, with numerical data (N_i, v_i), $i \in T$, for $\mathcal{I}_{\boldsymbol{f}}$. Then $Z_\Phi(s, \boldsymbol{f})$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$ and the poles are contained in the union of*

$$-\frac{v_i}{N_i} - \frac{\mathbb{N}}{N_i}, \ i \in T.$$

Therefore the poles are negative rational numbers. Moreover their orders are at most equal to n.

Proof. We use all the notations concerning the log-principalization h introduced in Theorem 1. Let $b \in X_K$ be a point, and (ϕ_V, V) a chart containing it. Let $E_1, \ldots, E_r$ denote the components of $h^{-1}\left(\boldsymbol{f}^{-1}(0)\right)$ passing through b. We set $\boldsymbol{f}^*(y) := \boldsymbol{f}(h(y))$.

If $r = 0$, i.e., $\boldsymbol{f}(h(b)) \neq 0$, then we can choose a small neighborhood V_b of b over which $|\boldsymbol{f}^*(y)|_K$ is positive and $\mathbb{R}$-analytic, and thus

(2.1) $\quad |\boldsymbol{f}^*(y)|_K^s = e^{s \ln |\boldsymbol{f}^*(y)|_K}$ is $\mathbb{R}$-analytic in $y \in V_b$ and holomorphic in $s \in \mathbb{C}$.

In addition,

$$h^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y)\,(dy_1 \wedge \ldots \wedge dy_n), \tag{2.2}$$

where $\eta(y)$ is a unit of the local ring of X_K at b.

If $r \geq 1$, then in V,

$$f_i^*(y) = f_i(h(y)) = g(y)\widetilde{f}_i(y),\ i = 1, \ldots, l, \tag{2.3}$$

$$g(y) = \varepsilon(y) \prod_{i=1}^{r} y_i^{N_i}, \tag{2.4}$$

and

$$h^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y) \prod_{i=1}^{r} y_i^{v_i - 1}(dy_1 \wedge \ldots \wedge dy_n), \tag{2.5}$$

where $\varepsilon(y)$ and $\eta(y)$ are units of the local ring of X_K at b. Furthermore, there exists an index i_0 such that $\widetilde{f}_{i_0}(b) \neq 0$. Then

$$|\boldsymbol{f}^*(y)|_K^s = |\varepsilon(y)|_K^s \left(\prod_{i=1}^{r} |y_i|_K^{N_i}\right)^s \left|\widetilde{\boldsymbol{f}}(y)\right|_K^s,$$

where $\widetilde{\boldsymbol{f}}(y) := \left(\widetilde{f}_1(y), \ldots, \widetilde{f}_l(y)\right)$, in V.

We can choose a small neighborhood V_b of b over which (2.3)-(2.5) are valid, and $|\varepsilon(y)|_K$, $\left|\widetilde{\boldsymbol{f}}(y)\right|_K$, $|\eta(y)|_K$ are $\mathbb{R}$-analytic. Then $|\varepsilon(y)|_K^s$, $\left|\widetilde{\boldsymbol{f}}(y)\right|_K^s$ are $\mathbb{R}$-analytic in y for any $s \in \mathbb{C}$, and holomorphic in $s \in \mathbb{C}$, for any $y \in V_b$.

Since $h^{-1}(Supp\ \Phi)$ is compact, we can take a finite covering of the form $\{V_b\}$ where the V_b are homeomorphic under ϕ_V to the polydisc $P_\epsilon(0)$ in K^n defined by $|y_i|_K < \epsilon$, with ϵ sufficiently small and for $1 \leq i \leq n$. By picking a smooth partition of the unity subordinate to $\{V_b\}$, and using the previous discussion,

$$Z_\Phi(s, \boldsymbol{f}) = \int_{X_K \setminus h^{-1}\left(\boldsymbol{f}^{-1}(0)\right)} \Phi^*(y)\, |\boldsymbol{f}^*(y)|_K^s\, |h^*(dx_1 \wedge \ldots \wedge dx_n)|$$

becomes a finite sum of integrals of the following two types:

$$I(s) := \int_{K^n} \Psi(y)\, |\boldsymbol{f}^*(y)|_K^s\, |dy|, \tag{2.6}$$

where Ψ is a C_0^∞ function with support contained in a polydisc $P_\epsilon(0)$ and $e^{s \ln |\boldsymbol{f}^*(y)|_K}$ is $\mathbb{R}-$analytic for $y \in V_b$ and holomorphic in $s \in \mathbb{C}$, or

$$J(s) := \int_{K^n} \Theta(y, s) \left(\prod_{i=1}^r |y_i|_K^{N_i s + v_i - 1}\right) |dy|, \tag{2.7}$$

where $\Theta(y, s)$ is a C_0^∞ function with support contained in a polydisc $P_\epsilon(0)$, depending holomorphically on $s \in \mathbb{C}$. By using the Dominated Convergence Lemma, we have that (2.6) defines a holomorphic function on the complex plane. The meromorphic continuation and the description of the corresponding poles for integrals $J(s)$ is known, see, for instance, the proofs of Theorem 5.4.1 in [13] or Theorem 1.6 in [11]. □

We indicate know why also the list of candidate poles in Theorem 2 for $K = \mathbb{R}$ can be derived from the function case. We think this proposition has some independent interest. A note on terminology: we still use the term log-principalization starting with one function, i.e. with a principal ideal; usually one calls this an embedded resolution.

Proposition 1. *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U \to \mathbb{R}^l$ be an $\mathbb{R}$–analytic mapping on an open $U \subseteq \mathbb{R}^n$. Then $h : X_\mathbb{R} \to U$ is a log-principalization of the ideal $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ if and only if it is a log-principalization of the function $F := \sum_{i=1}^l f_i^2$. Moreover, when $(N_i, v_i), i \in T$, are the numerical data of h for $\mathcal{I}_{\boldsymbol{f}}$, then $(2N_i, v_i), i \in T$, are the numerical data of h for F.*

Proof. We suppose that h is a log-principalization of the function F, with numerical data $(2N_i, v_i), i \in T$ ($2N_i$ will turn out to be even). We work in the local ring corresponding to some fixed point of $X_\mathbb{R}$; recall that this local ring is a unique factorization domain. Let $y_1, \ldots, y_n$ be coordinates at this point, i.e. a system of parameters of the local ring. We know that

$$F^*(y) = \sum_{i=1}^l (f_i^*(y))^2 = \epsilon(y) \prod_{i=1}^r y_i^{2N_i},$$

where $\epsilon(y)$ is a unit and $r \leq n$. Let $g(y) := \gcd_{1 \leq i \leq l} f_i^*(y)$ and write $f_i^*(y) = g(y)\widetilde{f}_i(y)$.

We claim that $\sum_{i=1}^{l}(\widetilde{f}_i(y))^2$ is a unit $u(y)$. Assuming the claim, we have that at least one of the $\widetilde{f}_i(y)$ is a unit. Hence

$$h^*\mathcal{I}_{\boldsymbol{f}} = (f_1^*(y), \dots, f_l^*(y)) = (g(y))$$

and $\epsilon(y)\prod_{i=1}^{r} y_i^{2N_i} = u(y)(g(y))^2$. So, up to a unit, $g(y)$ equals $\prod_{i=1}^{r} y_i^{N_i}$, and indeed h is a log-principalization of $\mathcal{I}_{\boldsymbol{f}}$.

We now prove the claim. We know that

$$\epsilon(y)\prod_{i=1}^{r} y_i^{2N_i} = (g(y))^2\left(\sum_{i=1}^{l}(\widetilde{f}_i(y))^2\right).$$

Since we have unique factorization, either this sum is a unit, or it is divisible by one of the irreducible elements of the left hand side, i.e. by y_1, y_2, …or y_r. Say $\sum_{i=1}^{l}(\widetilde{f}_i(y))^2$ is divisible by y_1. We can always write this sum as

$$\sum_{i=1}^{l}(\widetilde{f}_i(0, y_2, \dots, y_n))^2 + y_1(\dots).$$

Then divisibility by y_1 implies that $\sum_{i=1}^{l}(\widetilde{f}_i(0, y_2, \dots, y_n))^2 = 0$. Since we are working over $\mathbb{R}$ this can only happen if $\widetilde{f}_i(0, y_2, \dots, y_n) = 0$ for all $i = 1, \dots, l$. But this is equivalent to y_1 dividing all $\widetilde{f}_i(y_1, y_2, \dots, y_n)$, contradicting that the $\widetilde{f}_i(y)$ are relatively prime.

The other implication is quite straightforward. □

Remark 2. We recall the description of the (local and global) log canonical threshold, see e.g. [6]-[16]-[19], in terms of a log-principalization. Let $K = R$ or C and U an open submanifold of K^n. Let $I_{\boldsymbol{f}} = (f_1, \dots, f_l)$ be a non-trivial ideal of $K-$analytic functions on U. Fix a log-principalization $h : X_K \to U$ of $I_{\boldsymbol{f}}$ as in Theorem 1.
(1) Let $P \in f^{-1}(0)$. The (K-)log canonical threshold of $I_{\boldsymbol{f}}$ at P is $c_P(I_{\boldsymbol{f}}) = \min_{i\in T,\ P\in h(E_i)}\left\{\frac{v_i}{N_i}\right\}$.
(2) The (K-)log canonical threshold of $I_{\boldsymbol{f}}$ is $c(I_{\boldsymbol{f}}) = \min_{i\in T}\left\{\frac{v_i}{N_i}\right\}$.

Proposition 2. *Let $\boldsymbol{f}$ and Φ be as in 2.1.*
(1) Let $P \in \boldsymbol{f}^{-1}(0)$. If Φ is real and nonnegative with support in a small enough neighborhood of P (in particular $\Phi(P) > 0$), then $-c_P(\mathcal{I}_{\boldsymbol{f}})$ is a pole of $Z_\Phi(s, \boldsymbol{f})$, more precisely its largest pole.

Fix a log-principalization of $\mathcal{I}_{\boldsymbol{f}}$ as in Theorem 1.
(2) Say that $c(\mathcal{I}_{\boldsymbol{f}}) = \frac{v_i}{N_i}$ precisely for $i \in T_\lambda (\subset T)$. If Φ is real and nonnegative and its support intersects $h(\cup_{i\in T_\lambda} E_i)$, then $-c(\mathcal{I}_{\boldsymbol{f}})$ is a pole of $Z_\Phi(s, \boldsymbol{f})$, more precisely its largest pole.
(3) Let $r(\mathcal{I}_{\boldsymbol{f}})$ be the maximal number of $E_i, i \in T$, with $c(\mathcal{I}_{\boldsymbol{f}}) = \frac{v_i}{N_i}$, respectively $c_P(\mathcal{I}_{\boldsymbol{f}}) = \frac{v_i}{N_i}$, that have a nonempty intersection. Then $-c(\mathcal{I}_{\boldsymbol{f}})$, respectively $-c_P(\mathcal{I}_{\boldsymbol{f}})$, is a pole of order $r(\mathcal{I}_{\boldsymbol{f}})$ of $Z_\Phi(s, \boldsymbol{f})$ when Φ is as above.

Proof. One uses the case of monomial integrals like in the case of one analytic function, see e.g. [1, Chap. II, § 7]. The proof is a simple variation of the one given in [1, Chap. II, § 7, Lemme 4] and [12, pp. 32-33] for the case $l = 1$. For $K = \mathbb{R}$ one can alternatively *use* the case $l = 1$ and Proposition 1. □

Note that this proposition gives an argument to see that these minima do not depend on the chosen log-principalization.

Corollary 1. *Let K be $\mathbb{R}$ or $\mathbb{C}$ as before, and $\boldsymbol{f}$ as in 2.1. Let P be a compact subset of K^n. Then*

(1) $|\boldsymbol{f}(x)|_K^{\delta}$ is locally integrable for $\delta > -c(\mathcal{I}_{\boldsymbol{f}})$;

(2) Vol $(\{x \in P \mid |\boldsymbol{f}(x)|_K \leq \alpha\}) \leq \alpha^{c(\mathcal{I}_{\boldsymbol{f}})-\epsilon} \int_P |\boldsymbol{f}(x)|_K^{-c(\mathcal{I}_{\boldsymbol{f}})+\epsilon} |dx|$, for $\alpha > 0$ and any small $\epsilon > 0$.

Proof. The first part follows directly from Proposition 2. Then the second part follows via the Chebyshev inequality. □

Such bounds on volumes have recently emerged as central to aspects of complex differential geometry, see [18] and references therein.

3. Newton Polyhedra

We collect some results about Newton polyhedra and log-principalizations following [21] and the references therein. In this section we take again $K = \mathbb{R}$ or $K = \mathbb{C}$.

We set $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geqslant 0\}$. Let G be a nonempty subset of $\mathbb{N}^n$. The *Newton polyhedron* $\Gamma = \Gamma(G)$ associated to G is the convex hull in $\mathbb{R}_+^n$ of the set $\cup_{m\in G}\left(m + \mathbb{R}_+^n\right)$. For instance classically one associates a *Newton polyhedron (at the origin) to* $g(x) = \sum_m c_m x^m$ $(x = (x_1, \ldots, x_n),\ g(0) = 0)$, being a nonconstant polynomial function over K or a K–analytic function in a neighborhood of the origin, where $G = \operatorname{supp}(g) := \{m \in \mathbb{N}^n \mid c_m \neq 0\}$. Further we associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron Γ as above. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

For $a \in \mathbb{R}_+^n$, we define

$$d(a, \Gamma) = d(a) = \min_{x\in\Gamma} \langle a, x\rangle\,,$$

and *the first meet locus* $F(a)$ of a as

$$F(a) := \{x \in \Gamma \mid \langle a, x\rangle = d(a)\}.$$

The first meet locus is a face of Γ. Moreover, if $a \neq 0$, $F(a)$ is a proper face of Γ.

We define an equivalence relation in $\mathbb{R}_+^n$ by taking $a \sim a' \Leftrightarrow F(a) = F(a')$. The equivalence classes of $\sim$ are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a) = \tau\},$$

where τ is a face of Γ.

We recall that the cone strictly spanned by the vectors $a_1, \ldots, a_r \in \mathbb{R}_+^n \setminus \{0\}$ is the set $\Delta = \{\lambda_1 a_1 + ... + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_+,\ \lambda_i > 0\}$. If $a_1, \ldots, a_r$ are linearly independent over $\mathbb{R}$, Δ is called a *simplicial cone*. If $\{a_1, \ldots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ a *simple cone*.

A precise description of the geometry of the equivalence classes modulo $\sim$ is as follows. Each *facet* (i.e. a face of codimension one) γ of Γ has a unique vector $a(\gamma) = (a_{\gamma,1}, \ldots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$, whose nonzero coordinates are relatively prime, which is perpendicular to γ. We denote by $\mathfrak{D}(\Gamma)$ the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \{\sum_{i=1}^{r} \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+,\ \lambda_i > 0\},$$

where τ runs through the set of faces of Γ, and γ_i, $i = 1, \ldots, r$ are the facets containing τ. We note that $\Delta_\tau = \{0\}$ if and only if $\tau = \Gamma$. The family $\{\Delta_\tau\}_\tau$, with τ running over the proper faces of Γ, is a partition of $\mathbb{R}_+^n \backslash \{0\}$; we call this partition a *polyhedral subdivision of* $\mathbb{R}_+^n$ *subordinated to* Γ. We call $\left\{\overline{\Delta_\tau}\right\}_\tau$, the family formed by the topological closures of the Δ_τ, a *fan subordinated to* Γ.

Each cone Δ_τ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}$. In addition, the subdivision can be chosen such that each $\Delta_{\tau,i}$ is spanned by part of $\mathfrak{D}(\Gamma)$. Thus from the above considerations we have the following partition of $\mathbb{R}_+^n \backslash \{0\}$:

$$\mathbb{R}_+^n \backslash \{0\} = \bigcup_\tau \left(\bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right), \tag{3.1}$$

where τ runs over the proper faces of Γ, and each $\Delta_{\tau,i}$ is a simplicial cone contained in Δ_τ.

By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision* of $\mathbb{R}_+^n$ *subordinated to* Γ and a *simple fan subordinated to* Γ (see e.g. [14]).

3.1. **The Newton polyhedron associated to an analytic mapping.** Let $\boldsymbol{f} = (f_1, \ldots, f_l)$, $\boldsymbol{f}(0) = 0$, be a nonconstant analytic mapping defined on a neighborhood $U \subseteq K^n$ of the origin. In [21] the authors associated to $\boldsymbol{f}$ a Newton polyhedron $\Gamma(\boldsymbol{f}) := \Gamma\left(\cup_{i=1}^{l} \mathrm{supp}\,(f_i)\right)$, and a non-degeneracy condition to $\boldsymbol{f}$ and $\Gamma(\boldsymbol{f})$.

If $f_i(x) = \sum_m c_{m,i} x^m$, and τ is a face of $\Gamma(\boldsymbol{f})$, we set

$$f_{i,\tau}(x) := \sum_{m \in \mathrm{supp}(f_i) \cap \tau} c_{m,i} x^m.$$

Definition 1. Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U(\subseteq K^n) \longrightarrow K^l$ be a nonconstant analytic mapping satisfying $\boldsymbol{f}(0) = 0$. The mapping $\boldsymbol{f}$ is called *strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$, if for any *compact* face $\tau \subset \Gamma(\boldsymbol{f})$ and any $z \in \left\{z \in (K^\times)^n \mid f_{1,\tau}(z) = \ldots = f_{l,\tau}(z) = 0\right\}$ it verifies that

$$rank_K \left[\frac{\partial f_{i,\tau}}{\partial x_j}(z) \right] = \min\{l, n\}.$$

In other words, we require for each compact face τ that $0 \in K^l$ is not a critical value of the mapping $(f_{1,\tau}, \ldots, f_{l,\tau}) : (K^\times)^n \to K^l$.

Remark 3. (1) The above notion of non-degeneracy agrees with the one give by Varchenko for the case $l = 1$, see [20]. On the other hand, the previous notion does not agree with the non-degeneracy notion with respect to a collection of Newton polyhedra given by Khovanskii in [15]. We refer the reader to [21] for a further discussion about the mentioned non-degeneracy conditions.
(2) In [15], Khovanskii established the existence of an embedded resolution for a variety using a collection of Newton polyhedra, see also [1] for the case $l = 1$. In [21],

a log-principalization for an ideal with generators satisfying the above-mentioned notion of non-degeneracy and using one Newton polyhedron was established, see Proposition 3 below. This result agrees with Khovanskii's result only in the case $l = 1$.

Remark 4. When $K = \mathbb{R}$ it is again natural to try to study the mapping $f = (f_1, \ldots, f_l)$ via the function $F := \sum_{i=1}^{l} f_i^2$. It is not difficult to verify that $\Gamma(F)$ is the 'double' of $\Gamma(f)$, i.e. obtained from it after scaling by a factor 2. Consider however the statements

(i) F is non-degenerate at the origin with respect to $\Gamma(F)$, and

(ii) f is strongly non-degenerate at the origin with respect to $\Gamma(f)$.

It is easy to verify that (i) implies (ii), but in general the converse is not true. Consider for instance any strongly non-degenerate f for which $\{f_{1,\tau} = \ldots = f_{l,\tau} = 0\} \cap (\mathbb{R}^{\times})^n \neq \emptyset$ for some compact face τ of $\Gamma(f)$, e.g. $f = (x^2 - y^3, x^2 - z^3)$.

In such cases the 'classical' embedded resolution of a non-degenerate function is not helpful, but one can use the log-principalization of a strongly non-degenerate mapping of Proposition 3 below.

3.2. **Newton polyhedra and log-principalizations.**

Proposition 3 ([21, Prop. 3.9])**.** *Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U(\subseteq K^n) \longrightarrow K^l$ be a nonconstant analytic mapping, strongly non-degenerate at the origin with respect to $\Gamma(\boldsymbol{f})$. Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinated to $\Gamma(\boldsymbol{f})$. Let Y_K be the toric manifold corresponding to $\mathcal{F}_{\boldsymbol{f}}$, and let*

$$\sigma_0 : Y_K \longrightarrow U$$

be the restriction of the corresponding toric map to the inverse image of U. Denote by S the set of common zeroes of $\mathcal{I}_{\boldsymbol{f}} = (f_1, \ldots, f_l)$ in $U \cap (K^{\times})^n$. When U is taken small enough, either $S = \varnothing$ or it is a submanifold of codimension l. In this last case we have $l < n$ and we denote the closure of S in Y_K by S_Y.

(1) If $S = \varnothing$ (or if $l = 1$), the ideal $\sigma_0^*(\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma_0^{-1}\{0\}$.

(2) If $S \neq \varnothing$, we have that S_Y is a closed submanifold of Y_K, having normal crossings with the exceptional divisor of σ_0. Let $\sigma_1 : X_K \longrightarrow Y_K$ be the blowing-up of Y_K with center S_Y, and let $\sigma = \sigma_0 \circ \sigma_1 : X_K \longrightarrow U$. Then the ideal $\sigma^*(\mathcal{I}_{\boldsymbol{f}})$ is principal (and monomial) in a sufficiently small neighborhood of $\sigma^{-1}\{0\}$.

Given $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{N}^n \setminus \{0\}$, we put $\sigma(\xi) := \xi_1 + \ldots + \xi_n$ and $d(\xi) = \min_{x \in \Gamma(\boldsymbol{f})} \langle \xi, x \rangle$ as before. We say that ξ is a primitive vector, if $\gcd(\xi_1, \ldots, \xi_n) = 1$. If $d(\xi) \neq 0$, we define

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi) + k}{d(\xi)} \mid k \in \mathbb{N} \right\}.$$

We also define

$$\gamma_0(\boldsymbol{f}) = \min_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \left\{ \frac{\sigma(\xi)}{d(\xi)} \right\}.$$

Varchenko called $\gamma_0(\boldsymbol{f})$ the distance from the origin to $\Gamma(\boldsymbol{f})$. The number $\gamma_0(\boldsymbol{f})$ admits the following geometric interpretation. Let $(t_0, \ldots, t_0)$ be the intersection point of the diagonal $\{(t, \ldots, t) \in \mathbb{R}^n \mid t \in \mathbb{R}\}$ with the boundary of $\Gamma(\boldsymbol{f})$, then $\gamma_0(\boldsymbol{f}) = 1/t_0$. Let $\tau_{\mathbf{f}}$ be the smallest face of $\Gamma(\boldsymbol{f})$ containing $(t_0, \ldots, t_0)$. We set $\kappa_{\boldsymbol{f}}$ for the codimension of $\tau_{\mathbf{f}}$ in $\mathbb{R}^n$.

Let $\mathcal{F}_{\boldsymbol{f}}$ be a simple fan subordinated to $\Gamma(\boldsymbol{f})$. Then the set of generators of the cones in $\mathcal{F}_{\boldsymbol{f}}$, i.e. the skeleton of $\mathcal{F}_{\boldsymbol{f}}$, can be partitioned as $\Lambda_{\boldsymbol{f}} \cup \mathfrak{D}(\Gamma(\boldsymbol{f}))$, where $\Lambda_{\boldsymbol{f}}$ is a finite set of primitive vectors, corresponding to the extra rays, induced by the subdivision into simple cones.

The numerical data of the log-principalizations constructed in Proposition 3 can be computed directly from the explicit expressions for the generators of $\sigma_0^*(I_f)$, $\sigma^*(I_f)$, and Lemma 8 in [1, p. 201]. The following theorem follows from the previous considerations by adapting the proof given by Varchenko for the case $l=1$ to the case $l \geq 1$.

Theorem 3. *(1) The function* $Z_\Phi(s, \boldsymbol{f})$ *is holomorphic on* $\mathrm{Re}(s) > \max\{-\gamma_0(\boldsymbol{f}), -l\}$.
(2) Let $\boldsymbol{f} = (f_1, \ldots, f_l) : U(\subset K^n) \longrightarrow K^l$ *be an analytic mapping, strongly non-degenerate at the origin with respect to* $\Gamma(\boldsymbol{f})$*. If* U *is a sufficiently small neighborhood of the origin, and* Φ *is a smooth function whose support is contained in* U*, then the poles of* $Z_\Phi(s, \boldsymbol{f})$ *belong to the set* $\cup_{\xi \in \mathfrak{D}(\Gamma(\boldsymbol{f}))} \mathcal{P}(\xi) \cup \cup_{\xi \in \Lambda_{\boldsymbol{f}}} \mathcal{P}(\xi) \cup (-(l + \mathbb{N}))$*, where the last set may be discarded if* $l \geq n$.
(2) If $\gamma_0(\boldsymbol{f}) < l$*, then* $s = -\gamma_0(\boldsymbol{f})$ *is a pole of* $Z_\Phi(s, \boldsymbol{f})$ *as a distribution on the space of smooth functions with compact support.*

Remark 5. In the case $l = 1$ and $K = \mathbb{R}$, Denef and Sargos proved in [5] that $\cup_{\xi \in \Lambda_{\boldsymbol{f}}} P(\xi)$ may be discarded from the list of candidate poles in Theorem 3. This is a strong and interesting result, yielding in general a much shorter list of candidate poles, that can moreover be read off immediately from $\Gamma(\boldsymbol{f})$. We expect that we can generalize this result to arbitrary $l \geq 1$; this will be pursued elsewhere.

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