# MONODROMY EIGENVALUES ARE INDUCED BY POLES OF ZETA FUNCTIONS
## – THE IRREDUCIBLE CURVE CASE

ANDRÁS NÉMETHI AND WILLEM VEYS

Abstract. The 'monodromy conjecture' for a hypersurface singularity $f$ predicts that a pole of its topological (or related) zeta function induces one of its monodromy eigenvalues. However, in general only a few eigenvalues are obtained this way. The second author proposed to consider zeta functions associated with the hypersurface and with a differential form and raised the following question. Can one find a list of differential forms $\omega_i$ such that any pole of the zeta function of $f$ and an $\omega_i$ induces a monodromy eigenvalue of $f$, and such that all monodromy eigenvalues of $f$ are obtained this way? Here we provide an affirmative answer for an arbitrary irreducible curve singularity $f$.

## 1. Introduction

1.1. The 'Monodromy Conjecture' is one of the most fertilizing conjectures in singularity theory. It relates poles of Igusa/motivic/topological zeta functions to monodromy eigenvalues. For instance for a local analytic isolated singularity $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ it predicts that if $s_0$ is a pole of the local topological zeta function of $f$ then $\lambda = \exp(2\pi i s_0)$ is an eigenvalue of the local monodromy operator acting on $H^*(F_0, \mathbb{C})$, where $F_0$ is the Milnor fiber of $f$. This conjecture was proved for $n = 2$ by Loeser (originally in the context of $p$-adic Igusa zeta functions) in [11]. There are by now various other partial results, e.g. [2, 3, 10, 12, 14, 17].

It is easy to see on explicit examples that in this way not all the eigenvalues of the monodromy operator are realized; actually quite few eigenvalues are obtained this way (in general). Hence it is natural to try to extend in some way this set of poles, such that the same procedure would yield all eigenvalues. A natural way to do this is using the local topological zeta functions associated with the original germ $f$ and with a set of analytic differential $n$-forms $\omega$ living in $(\mathbb{C}^n, 0)$.

We now first describe these zeta functions; they are defined in terms of an embedded resolution $\pi$ of $f^{-1}\{0\} \cup \operatorname{div}(\omega)$. We denote by $E_i, i \in S$, the irreducible components of the inverse image $\pi^{-1}(f^{-1}\{0\} \cup \operatorname{div}(\omega))$ and by $N_i$ and $\nu_i - 1$ the multiplicities of $E_i$ in the divisor of $\pi^* f$ and $\pi^*\omega$, respectively. We put $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{j \notin I} E_j)$ for $I \subset S$. So the $E_I^\circ$ form a stratification of $X$ in locally closed subsets.

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**Definition 1.1.1.** *The local topological zeta function of* $(f,\omega)$ *at* $0\in\mathbb{C}^n$ *is*

$$Z(f,\omega;s):=\sum_{I\subset S}\chi(E_I^\circ\cap\pi^{-1}\{0\})\prod_{i\in I}\frac{1}{\nu_i+sN_i},$$

*where* $s$ *is a variable.*

This invariant was introduced by Denef and Loeser in [6] for 'trivial $\omega$', i.e. for $\omega = dx_1\wedge\cdots\wedge dx_n$. Their original proof that this expression does not depend on the chosen resolution is by describing them as a kind of limit of $p$-adic Igusa zeta functions. Later they obtained the statement as a specialization of the intrinsically defined motivic zeta functions [7]. Another technique is applying the Weak Factorization Theorem [4], [19] to compare two different resolutions. For arbitrary $\omega$ one can proceed analogously.

It is natural and useful to study these invariants incorporating such a more general $\omega$, see for example [2, 3, 16]. Note however that there one restricts to the situation where $supp(\operatorname{div}(\omega))\subset f^{-1}\{0\}$. In the original context of $p$-adic Igusa zeta functions, see e.g. [11, III 3.5].

1.2. The second author in [18] proved for arbitrary $n$ that any given monodromy eigenvalue of $f$ at 0 is induced by a pole of a local topological zeta function of $f$ and an adequate $\omega$. In particular for $n=2$ the result is the following.

**Theorem 1.2.1.** *Assume that* $n=2$ *and let* $\lambda$ *be a monodromy eigenvalue of* $f$ *at* 0. *Then there exists a differential 2-form* $\omega$ *on* $(\mathbb{C}^2,0)$ *such that* $Z(f,\omega;s)$ *has a pole* $s_0$ *satisfying* $\exp(2\pi i s_0)=\lambda$.

On the other hand, the zeta functions $Z(f,\omega;s)$ constructed in the theorem above will in general have other poles that *don't* induce monodromy eigenvalues of $f$. The second author in [18] posed the problem to investigate the validity of the following statement:

**Problem 1.2.2.** Let $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ be a non-zero singularity germ. Find differential 2-forms $\omega_1,\cdots,\omega_k$ on $(\mathbb{C}^2,0)$ such that

(1) if $s_0$ is a pole of a zeta function $Z(f,\omega_j;s)$, then $\exp(2\pi i s_0)$ is a monodromy eigenvalue of $f$ at 0, and

(2) for each monodromy eigenvalue $\lambda$ of $f$ at 0, there is a differential form $\omega_j$ in the above list and a pole $s_0$ of $Z(f,\omega_j;s)$ such that $\exp(2\pi i s_0)=\lambda$.

(A similar problem can be stated for arbitrary $n$.) As an illustration the case $f=y^q-x^p$ with $\gcd(p,q)=1$ was treated: for this $f$ the forms $x^{i-1}y^{j-1}dx\wedge dy$ with $1\leq i\leq p-1$ and $1\leq j\leq q-1$ satisfy (1) and (2). However, the 'simplicity' of this case can be misleading. When one constructs an $\omega_j$ as in (2) to induce a given $\lambda$, the zeta function $Z(f,\omega_j;s)$ will typically have other poles inducing roots of unity that are *not* eigenvalues.

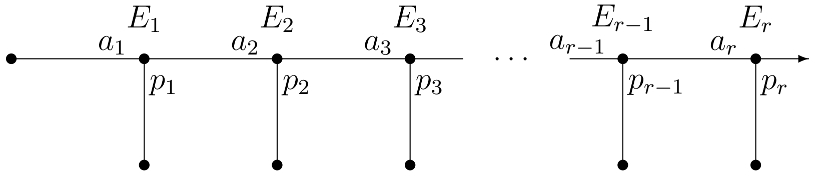
1.3. As a motivation for this type of problem we should mention that the same one can be stated for the $p$-adic Igusa zeta function, which is a certain $p$-adic integral. For similar complex integrals there are (for arbitrary $n$) theorems of Barlet [5], Kashiwara [9] and Malgrange [13] relating poles to monodromy eigenvalues. We refer to the introduction of [18] for more details.

1.4. The main result of the present article is to give a positive answer to the above Problem in the case when $f$ is an arbitrary (analytically) irreducible singularity.

## 2. Preliminaries

We fix the notation that we will use throughout the paper for the invariants of an irreducible complex plane curve singularity $\{f=0\}$. We recall some well known facts about them, and we state an efficient compact formula for the topological zeta function of $f$ and the relevant differential forms $\omega$.

2.1. Let $\{f=0\}$ be an irreducible curve singularity with $r$ Puiseux pairs ($r\geq 1$). The Eisenbud-Neumann, or splice diagram $\Gamma$ of $f$ is the following:



A way to look at it is as the dual graph $G$ of the minimal embedded resolution $\pi:(X,E)\to(\mathbb{C}^2,0)$ of $\{f=0\}$, *where the vertices of valence 2 are deleted.* The vertices $E_1,\cdots,E_r$ correspond to the so-called rupture irreducible components of the exceptional locus $E$, i.e. those intersecting at least three times other components. The arrow corresponds to the strict transform of $\{f=0\}$. For details regarding splice diagrams, see [8].

The splice decorations satisfy $a_\ell\geq 2$ and $p_\ell\geq 2$. Their meaning is the following. They are (the absolute value of) the determinant of the self-intersection matrix of *all* exceptional components of $\pi$, appearing in $G$ away from $E_\ell$ in the direction of $a_\ell$, resp. $p_\ell$. In fact, they can be recovered from the Newton pairs $(p_\ell,q_\ell)_{\ell=1}^r$ of $f$ by the following formula:

$$\begin{cases} a_1 = q_1\\ a_2 = q_2 + a_1p_1p_2\\ a_3 = q_3 + a_2p_2p_3\\ \cdots\\ a_r = q_r + a_{r-1}p_{r-1}p_r. \end{cases} \tag{2.1.1}$$

Above $q_\ell\geq 1$ and $\gcd(p_\ell,q_\ell)=1$. Moreover, $q_\ell$ is also the absolute value of the determinant of the self-intersection matrix of all exceptional components on the chain strictly between $E_{\ell-1}$ and $E_\ell$, where we take $q_\ell=1$ if there are no such components (i.e. if $E_{\ell-1}$ intersects $E_\ell$). In particular, we have for all $\ell=2,\cdots,r$

$$a_\ell > a_1p_1p_2^2\cdots p_{\ell-1}^2p_\ell. \tag{2.1.2}$$

2.2. Let $F_0$ denote the local Milnor fibre of $f$ at 0. Then the characteristic polynomial $P(f;t)$ of the local monodromy action on the first cohomology $H^1(F_0,\mathbb{C})$ is

$$\Delta_{a_1,p_1}(t^{p_2p_3\cdots p_r})\cdot\Delta_{a_2,p_2}(t^{p_3\cdots p_r})\cdot\ldots\cdot\Delta_{a_{r-1},p_{r-1}}(t^{p_r})\cdot\Delta_{a_r,p_r}(t),$$

where we use the notation $\Delta_{c,d}(t):=\frac{(t^{cd}-1)(t-1)}{(t^c-1)(t^d-1)}\in\mathbb{Z}[t]$.

2.3. For $\ell = 1, \cdots, r$, let $N_\ell$ and $\nu_\ell - 1$ be the multiplicities of $E_\ell$ in the divisor of $\pi^* f$ and $\pi^*(dx \wedge dy)$, respectively.

**Lemma 2.3.1.**

$$\begin{cases} N_\ell = a_\ell p_\ell p_{\ell+1} \cdots p_r \\ \nu_\ell = p_\ell \nu_{\ell-1} + q_\ell, \ \ \textit{where } \nu_0 = 1. \end{cases}$$

*Proof.* The first identity is a well-known multiplicity formula of splice diagrams: $N_\ell$ is the product of all decorations along but not on the path connecting $E_\ell$ with the arrowhead [8, §10]. For the second part, consider for a moment the minimal resolution graph with deleted arrowhead, denoted by $G'$. Let $\{F_k\}_k$ be the set of exceptional divisors/vertices, and $\delta_k$ the valency of $F_k$ in $G'$. If $K$ is the canonical cycle on $G'$ (or on $X$), and $F := \sum_k F_k$, then by the adjunction formula $K + F = \sum_k (\delta_k - 2) F_k^*$, where $F_k^*$ is the dual cycle of $F_k$ with respect to the intersection form $( \, , \, )$ associated with $G'$. Since $F_k$ with $\delta_k \neq 2$ can be identified with vertices of $\Gamma$, one has

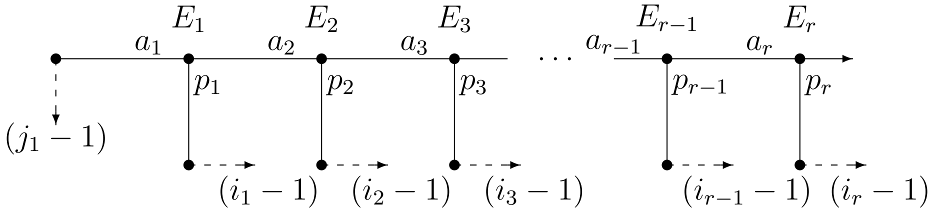
$$\nu_\ell = \sum_k (\delta_k - 2)(F_k^*, E_\ell^*).$$

The point is that for $\delta_k \neq 2$ the integer $-(F_k^*, E_\ell^*)$ can be read from $\Gamma$ too, it is the product of all the decorations along but not on the path connecting $F_k$ with $E_\ell$, see [8, §10 and §20]. Hence $\nu_\ell$ can be computed explicitly in $G'$, and the inductive formula also follows. ☐

For $I = (j_1, i_1, i_2, \cdots, i_r) \in (\mathbb{Z}_{>0})^{r+1}$ we will consider differential forms

$$\omega_I := g^{j_1 - 1} \prod_{\ell=1}^{r} g_\ell^{i_\ell - 1} dx \wedge dy.$$

Here $g_\ell = 0$ and $g = 0$ are the (reduced) equations of irreducible curves, whose strict transforms intersect exactly one exceptional component as indicated below (these strict transforms are represented by dashed arrows). We denote this graph by $\Gamma(I)$.



For $\ell \in \{1, \cdots, r\}$, let $\nu_{\ell,I} - 1$ denote the multiplicity of $E_\ell$ in the divisor of $\pi^*(\omega_I)$. Then by a similar multiplicity computation as in the proof of (2.3.1), we get:

$$\begin{aligned} \nu_{\ell,I} = \nu_\ell &+ p_\ell \cdots p_2 p_1 (j_1 - 1) + p_\ell \cdots p_2 a_1 (i_1 - 1) \\ &+ p_\ell \cdots p_3 a_2 (i_2 - 1) + p_\ell \cdots p_4 a_3 (i_3 - 1) + \cdots \\ &+ p_\ell a_{\ell-1} (i_{\ell-1} - 1) + a_\ell (i_\ell - 1) + p_\ell a_\ell (i_{\ell+1} - 1) \\ &+ p_{\ell+1} p_\ell a_\ell (i_{\ell+2} - 1) + \cdots + p_{r-1} \cdots p_\ell a_\ell (i_r - 1). \end{aligned}$$

2.4. Using [15] we can write a compact formula for $Z(f,\omega_I;s)$, associating only a term to each vertex $E_\ell$, to the straight arrow and to the edges in the graph $\Gamma(I)$.

**Proposition 2.4.1.**

$$\begin{aligned}Z(f,\omega_I;s) =&\frac{a_1}{j_1(\nu_{1,I}+sN_1)}+\sum_{\ell=1}^{r}\frac{-1}{\nu_{\ell,I}+sN_\ell}+\sum_{\ell=2}^{r}\frac{q_\ell}{(\nu_{\ell-1,I}+sN_{\ell-1})(\nu_{\ell,I}+sN_\ell)}\\ &+\frac{1}{(\nu_{r,I}+sN_r)(1+s)}+\sum_{\ell=1}^{r}\frac{p_\ell}{i_\ell(\nu_{\ell,I}+sN_\ell)}\\ =&\frac{1}{\nu_{1,I}+sN_1}(-1+\frac{a_1}{j_1}+\frac{p_1}{i_1})+\sum_{\ell=2}^{r}\frac{1}{\nu_{\ell,I}+sN_\ell}(-1+\frac{q_\ell}{\nu_{\ell-1,I}+sN_{\ell-1}}+\frac{p_\ell}{i_\ell})\\ &+\frac{1}{(\nu_{r,I}+sN_r)(1+s)}.\end{aligned}$$

*Proof.* Essentially, this is an application of [15, Theorem 3.3]. In fact, that theorem is only treated in the context of the standard differential form $dx\wedge dy$ (so for $\omega_I$ with $I=(1,1,\cdots,1)$ ), but it is straightforward to verify that analogously the formula is valid for arbitrary $\omega_I, I=(j_1,i_1,\cdots,i_r)$. □

**Corollary 2.4.2.** *Fix $\ell\in\{1,\cdots,r\}$. Then $-\frac{\nu_{\ell,I}}{N_\ell}$ is a pole of order two of $Z(f,\omega_I;s)$ if and only if*

$$\frac{\nu_{\ell,I}}{N_\ell}=\frac{\nu_{\ell-1,I}}{N_{\ell-1}}\quad or\quad \frac{\nu_{\ell,I}}{N_\ell}=\frac{\nu_{\ell+1,I}}{N_{\ell+1}},$$

*where the first condition is empty for $\ell=1$, and we put $(\nu_{r+1,I},N_{r+1}):=(1,1)$.*

2.5. Notice that both $P(f;t)$ and $Z(f,\omega_I;s)$ depend only on the diagram $\Gamma(I)$, and not on the particular choice of the equations of $f$, $g_\ell$ and $g$. In particular, all the next arguments regarding these invariants depend merely on this graph.

## 3. The main theorem and its proof

3.1. The goal of the present article is to find a set $\mathcal{I}$ of indices $I$ (or set of differential forms $\omega_I$) such that

(1) *if $s_0$ is a pole of some $Z(f,\omega_I;s)$ ($I\in\mathcal{I}$), then* $\exp(2\pi i s_0)$ *is an eigenvalue of monodromy of $f$, and*

(2) *if $\lambda$ is an eigenvalue of monodromy of $f$, then there is such a form $\omega_I$ ($I\in\mathcal{I}$) and a pole $s_0$ of $Z(f,\omega_I;s)$ such that* $\exp(2\pi i s_0)=\lambda$.

The size of $\mathcal{I}$ is obstructed by both conditions (1) and (2). A larger set is obstructed more by (1), while if $\mathcal{I}$ is too small then it may not realize in (2) all the eigenvalues. Therefore, besides the existence of such a set, one may also ask how large/small it can be.

**Theorem 3.1.1.** *Let $f$ be an irreducible plane curve singularity with $r$ Puiseux pairs as above. Then there exist an infinite set $\mathcal{I}$, and a finite subset $\mathcal{J}$ of $\mathcal{I}$ of cardinality $(a_r-1)(p_r-1)$, such that both sets $\mathcal{I}$ and $\mathcal{J}$ satisfy conditions (1) and (2) above.*

**Remark 3.1.2.** (a) Statement (1) is obvious for the 'trivial' candidate pole $s_0 = -1$ since $1 = \exp(2\pi i(-1))$ is always the (unique) monodromy eigenvalue on $H^0(F_0, \mathbb{C})$. So further we can concentrate on candidate poles ($\neq -1$) induced by the exceptional curves $E_1, \cdots, E_r$. Concerning (2) for $\lambda = 1$, we will see that at least one constructed $\omega_I$ really has $s_0 = -1$ as pole.

(b) For $r = 1$ the cardinality of $\mathcal{J}$ is sharp (it is the smallest possible), provided that we deal with differential forms used in the present article. In fact, it is exactly the Milnor number $\mu(f)$ of the germ. Nevertheless, for large $r$, we believe that the size of $\mathcal{J}$ can be decreased even more (even with the present forms). It was not our goal to find the best minimal $\#\mathcal{J}$, but to find (for $r > 1$) a pair $\mathcal{J} \subset \mathcal{I}$ with $\#\mathcal{J} < \mu(f) < \#\mathcal{I}$ in order to emphasize that the cardinality of an index-set satisfying the combinatorial conditions (1) and (2) is not really related with the Milnor number of $f$ (at least not in this combinatorial way).

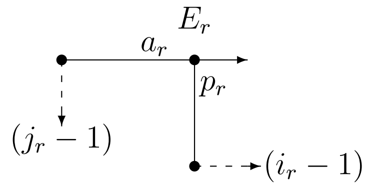
The proof of (3.1.1) will occupy the remaining part of the article.

3.2. **The inductive step.** The construction and the proof run over induction on the number of Puiseux pairs, i.e. sub-diagrams of $\Gamma$ with less and less rupture vertices. In order to compare nicely the corresponding invariants, the diagram with one Puiseux pair will be (strangely enough) the graph $\Gamma_{r,r}$ with one rupture vertex corresponding to $E_r$ and with splice decorations $(a_r, p_r)$. For any $1 \leq k \leq r$ we consider the diagram $\Gamma_{k,r}$ with rupture vertices corresponding to $\{E_\ell\}_{k \leq \ell \leq r}$ and splice decorations $\{(a_\ell, p_\ell)\}_{k \leq \ell \leq r}$. They codify the topological type of an irreducible germ $f_{k,r}$. Clearly, for $k = 1$ we recover $\Gamma_{1,r} = \Gamma$, $f_{1,r} = f$ (up to equisingularity).

Similarly, for an index set $\mathcal{I}_{k,r}$ of indices $I_{k,r} = (j_k, i_k, \ldots, i_r) \in (\mathbb{Z}_{>0})^{r-k+2}$ we consider the (topological/numerical type of the) form $\omega_{I_{k,r}}$ and the corresponding topological zeta function $Z(f_{k,r}, \omega_{I_{k,r}}; s)$.

In particular, the construction and the proof is based on a decreasing induction over $k$, and it starts with the diagram

$\Gamma_{r,r}(I_{r,r})$ :



3.3. **The construction of the sets $\mathcal{J} \subset \mathcal{I}$.** The sets $\mathcal{J}_{k,r} \subset \mathcal{I}_{k,r}$ will be constructed inductively as follows. We start with $k = r$:

$$\mathcal{I}_{r,r} := \{(j_r, i_r) \in (\mathbb{Z}_{>0})^2 \, : \, j_r \notin \{2a_r, 3a_r, \ldots\}, i_r \notin \{2p_r, 3p_r, \ldots\} \, \},$$

$$\mathcal{J}_{r,r} := \{(j_r, i_r) \, : \, 0 < j_r < a_r, \; 0 < i_r < p_r \, \}.$$

Assume that the sets $\mathcal{J}_{k+1,r} \subset \mathcal{I}_{k+1,r}$ are already constructed for some $k < r$. Then one defines a map

$$\psi_k : \mathcal{I}_{k+1,r} \to (\mathbb{Z}_{>0})^{r-k+2}$$

such that $\psi_k((j_{k+1}, i_{k+1}, \dots, i_r))$ has the form $(j_k, i_k, i_{k+1}, \dots, i_r)$ satisfying the additional identity

$$j_{k+1} = p_k j_k + a_k i_k - p_k a_k. \tag{3.3.1}$$

The pair $(j_k, i_k)$ is defined by the following principle:

**Case A:** $j_{k+1} \not\equiv 0 \pmod{p_k}$ and $j_{k+1} \not\equiv 0 \pmod{a_k}$.

Note that this is equivalent to '$i_k \not\equiv 0 \pmod{p_k}$ and $j_k \not\equiv 0 \pmod{a_k}$'. Since $\gcd(p_k, a_k) = 1$ the numbers $p_k x + a_k y$ for $x, y \in \mathbb{Z}_{>0}$ fill up all integers larger than $p_k a_k$. So, for a given $j_{k+1} \geq 1$ there exist $i_k \geq 1$ and $j_k \geq 1$ satisfying (3.3.1). We take such a pair $(j_k, i_k)$.

**Case B:** $j_{k+1} \not\equiv 0 \pmod{p_k}$ and $j_{k+1} \equiv 0 \pmod{a_k}$.

Take $j_k = a_k$ and $i_k = \frac{j_{k+1}}{a_k}$.

**Case C:** $j_{k+1} \equiv 0 \pmod{p_k}$.

Take $j_k = \frac{j_{k+1}}{p_k}$ and $i_k = p_k$.

**Definition 3.3.2.** *Set* $\mathcal{I}_{k,r} := \psi_k(\mathcal{I}_{k+1,r})$ *and* $\mathcal{J}_{k,r} := \psi_k(\mathcal{J}_{k+1,r})$ *for all* $k \in \{1, \dots, r-1\}$.

It is easy to verify that $\psi_k$ is injective, hence $\#\mathcal{J}_{k,r} = \#\mathcal{J}_{r,r} = (a_r - 1)(p_r - 1)$. In the main theorem we will take $\mathcal{I} := \mathcal{I}_{1,r}$ and $\mathcal{J} := \mathcal{J}_{1,r}$.

3.4. **The proof of (3.1.1) in the case of one Puiseux pair.** (This is essentially Proposition 2.7 in [18].) Consider the diagram $\Gamma_{r,r}(I_{r,r})$ from (3.2), where $I_{r,r} = (j_r, i_r) \in \mathcal{I}_{r,r}$ as in (3.3).

By (2.3.1) we have $N_r = a_r p_r$ and $\nu_{r,I} = \nu_r + p_r(j_r - 1) + a_r(i_r - 1) = p_r j_r + a_r i_r$. Using the formula of (2.4.1) we obtain

$$\begin{aligned} Z(f_{r,r}, \omega_{I_{r,r}}; s) =& \frac{1}{(p_r j_r + a_r i_r) + s a_r p_r} \cdot \Big( \frac{a_r}{j_r} + \frac{p_r}{i_r} + \frac{1}{1+s} - 1 \Big) \\ =& \frac{(p_r j_r + a_r i_r) + (p_r j_r + a_r i_r - j_r i_r)s}{j_r i_r (p_r j_r + a_r i_r + s a_r p_r)(1+s)}. \end{aligned}$$

First notice that $-1$ is always a pole, hence it can be paired with the trivial eigenvalue of the monodromy acting on $H^0(F_0, \mathbb{C})$. The candidate pole $s_0 := -\frac{p_r j_r + a_r i_r}{a_r p_r} = -(\frac{j_r}{a_r} + \frac{i_r}{p_r})$ cannot be a pole of order 2, since one easily verifies that $s_0 \neq -1$. It is a pole (of order 1) if and only if $p_r j_r + a_r i_r - j_r i_r \neq a_r p_r$, i.e. if and only if $i_r \neq p_r$ and $j_r \neq a_r$. Since in this case

$$P(f_{r,r}; t) = \Delta_{a_r, p_r}(t) = \frac{(t^{a_r p_r} - 1)(t - 1)}{(t^{a_r} - 1)(t^{p_r} - 1)},$$

the statements (1) and (2) are clearly satisfied for both $\mathcal{J}_{r,r}$ and $\mathcal{I}_{r,r}$.

3.5. **Inductive properties of the candidate poles.** In order to simplify the notations, we will consider the case $k = 1$ (but evidently, the same inductive formulae hold for arbitrary $k$ when one connects the steps $k+1$ with $k$). Moreover, we write $I' := I_{2,r} = (j_2, i_2, \dots, i_r)$ and $I := \psi_1(I') = (j_1, i_1, \dots, i_r)$. Then $\nu_{\ell,I}$ and $\nu_\ell$ (for any $1 \le \ell \le r$) are computed from $\Gamma_{1,r}(I)$, while their analogues $\nu_{\ell,I'}$ and $\nu'_\ell$ (for any $2 \le \ell \le r$) from $\Gamma_{2,r}(I')$.

In fact, with the substitution $j_2 = p_1 j_1 + a_1 i_1 - p_1 a_1$, one has the expressions

$$\begin{cases} \nu_{1,I} = p_1 j_1 + a_1 i_1 + a_1 p_1 (i_2 - 1) + a_1 p_1 p_2 [i_3 - 1 + p_3(i_4 - 1) + \cdots] \\ \quad \nu_{2,I} = p_2 j_2 + a_2 i_2 + a_2 p_2 [i_3 - 1 + p_3 (i_4 - 1) + \cdots], \end{cases}$$

where the terms between square brackets are the same. Therefore

$$a_2 \nu_{1,I} - a_1 p_1 \nu_{2,I} = j_2 (a_2 - a_1 p_1 p_2) = j_2 q_2. \tag{3.5.1}$$

Using this identity as inductive step, we obtain for all $\ell \in \{1, \dots, r-1\}$:

$$a_{\ell+1} \nu_{\ell,I} - a_\ell p_\ell \nu_{\ell+1,I} = j_{\ell+1} q_{\ell+1}. \tag{3.5.2}$$

Notice also that $N_\ell$ for $\ell \ge 2$ computed from $\Gamma_{1,r}$ or from $\Gamma_{2,r}$ is the same.

We list now several facts regarding the candidate poles $-\frac{\nu_{\ell,I}}{N_\ell}$ $(\ell \ge 1)$.

**Lemma 3.5.3.**

$$\frac{\nu_{1,I}}{N_1} > \frac{\nu_{2,I}}{N_2} > \cdots > \frac{\nu_{r,I}}{N_r}.$$

*In particular,* $-\frac{\nu_{1,I}}{N_1}$ *is not a pole of order 2 of* $Z(f_{1,r}, \omega_I; s)$.

*Proof.* For the inequalities use (3.5.2), then apply (2.4.2). □

We now compute the contribution $\mathcal{R}_1$ of $E_1$ to the residue of $-\frac{\nu_{1,I}}{N_1}$ as candidate pole of order 1 of $Z(f_{1,r}, \omega_I; s)$. Namely, by (2.4.1),

$$N_1 \mathcal{R}_1 = -1 + \frac{a_1}{j_1} + \frac{p_1}{i_1} + \frac{q_2}{\nu_{2,I} - \frac{\nu_{1,I}}{N_1} N_2} = -1 + \frac{a_1}{j_1} + \frac{p_1}{i_1} + \frac{a_1 p_1 q_2}{a_1 p_1 \nu_{2,I} - a_2 \nu_{1,I}}.$$

Note that this *is* the residue, unless $\frac{\nu_{1,I}}{N_1} = 1$.

**Lemma 3.5.4.** $\mathcal{R}_1 = 0$ *if and only if* $j_1 = a_1$ *or* $i_1 = p_1$.

*Proof.* Use identity (3.5.1). □

**Lemma 3.5.5.** *For* $\ell \ge 2$ *one has:*

(i) $\nu_{\ell,I} = \nu_{\ell,I'}$, *and*

(ii) $s_0 = -\frac{\nu_{\ell,I}}{N_\ell} = -\frac{\nu_{\ell,I'}}{N_\ell}$ *is a pole of* $Z(f_{1,r}, \omega_I; s)$ *if and only if it is a pole of* $Z(f_{2,r}, \omega_{I'}; s)$.

*Proof.* By the general formula

$$\begin{cases} \nu_{\ell,I'} = \nu'_\ell + p_\ell \cdots p_2 (j_2 - 1) + [p_\ell \cdots p_3 a_2 (i_2 - 1) + \cdots] \\ \nu_{\ell,I} = \nu_\ell + p_\ell \cdots p_2 p_1 (j_1 - 1) + p_\ell \cdots p_2 a_1 (i_1 - 1) + [p_\ell \cdots p_3 a_2 (i_2 - 1) + \cdots] \end{cases}$$

where the terms between square brackets are the same. Also, one easily verifies by induction on $\ell$ that $\nu'_\ell = \nu_\ell + p_\ell \cdots p_2 (p_1 - 1)(a_1 - 1)$. Hence (i) follows.

For (ii) notice that for $\ell \geq 3$ the relevant parts of the two zeta functions are exactly the same. For $\ell = 2$ again most relevant parts are the same; the difference is the contribution $\frac{q_2}{(\nu_{1,I}+sN_1)(\nu_{2,I}+sN_2)}$ for $Z(f_{1,r}, \omega_I; s)$ versus the contribution $\frac{a_2}{j_2(\nu_{2,I}+sN_2)}$ for $Z(f_{2,r}, \omega_{I'}; s)$. In order to check the claim about $s_0$ being a pole we must verify that $\frac{q_2}{\nu_{1,I}+s_0N_1} = \frac{a_2}{j_2}$. This is equivalent to $q_2 j_2 = a_2(\nu_{1,I} - \frac{\nu_{2,I}}{N_2} N_1) = a_2\nu_{1,I} - a_1 p_1 \nu_{2,I}$, which is precisely (3.5.1). ☐

Before we start the proof, we need another property of the set $\mathcal{J} = \mathcal{J}_{1,r}$.

**Lemma 3.5.6.** *The set* $\mathcal{J}$ *contains a subset* $\mathcal{J}'$ *such that the set of possible values* $\{\nu_{1,I} : I \in \mathcal{J}'\}$ *contains all the integers* $a_1p_1\cdots p_{r-1}+1, \ldots, a_1p_1\cdots p_{r-1}+N_1$.

*Proof.* We construct the desired $\mathcal{J}' = \mathcal{J}'_{1,r}$ inductively as follows. Set

$$\mathcal{J}'_{r,r} = \{(j_r, i_r) : 1 \leq i_r \leq p_r - 1,\ j_r = \alpha p_2 \cdots p_{r-1} \text{ for some } 1 \leq \alpha \leq N_1\}.$$

Since $N_1 p_2 \cdots p_{r-1} = a_1 p_1 p_2^2 \cdots p_{r-1}^2 p_r \leq a_r - 1$, cf. (2.1.2), $\mathcal{J}'_{r,r} \subset \mathcal{J}_{r,r}$. Define inductively $\mathcal{J}'_{k,r} := \psi_k(\mathcal{J}'_{k+1,r})$ for all $1 \leq k < r$.

Take $(j_r, i_r) \in \mathcal{J}'_{r,r}$. Then $\psi_{r-1}((j_r, i_r)) = (j_r/p_{r-1}, p_{r-1}, i_r)$, and inductively

$$\psi_2 \circ \cdots \circ \psi_{r-1}((j_r, i_r)) = (\alpha, p_2, \cdots, p_{r-1}, i_r).$$

Therefore, any $I \in \mathcal{J}'_{1,r}$ has the form $(j_1, i_1, p_2, \cdots, p_{r-1}, i_r)$ with $\alpha = p_1 j_1 + a_1 i_1 - p_1 a_1$, cf. (3.3.1). Hence, we have

$$\begin{aligned}\nu_{1,I} =& p_1 j_1 + a_1 i_1 + a_1 p_1 (i_2 - 1) + a_1 p_1 p_2 (i_3 - 1) + a_1 p_1 p_2 p_3 (i_4 - 1) \\ & + \cdots + a_1 p_1 p_2 \cdots p_{r-2}(i_{r-1} - 1) + a_1 p_1 p_2 \cdots p_{r-1}(i_r - 1) \\ =& \alpha + a_1 p_1 [p_2 + p_2(p_3 - 1) + p_2 p_3 (p_4 - 1) \\ & + \cdots + p_2 p_3 \cdots p_{r-2}(p_{r-1} - 1) + p_2 p_3 \cdots p_{r-1}(i_r - 1)] \\ =& \alpha + (a_1 p_1 p_2 \cdots p_{r-1}) i_r.\end{aligned}$$

Since $1 \leq \alpha \leq a_1 p_1 p_2 \cdots p_r$ and $1 \leq i_r \leq p_r - 1$ are arbitrary, these $\nu_{1,I}$ contain all the integers $1 + a_1 p_1 \cdots p_{r-1}, \ldots, a_1 p_1 \cdots p_r + a_1 p_1 \cdots p_{r-1}(p_r - 1)$. Moreover

$$a_1 p_1 \cdots p_r + a_1 p_1 \cdots p_{r-1}(p_r - 1) \geq a_1 p_1 \cdots p_r + a_1 p_1 \cdots p_{r-1} = N_1 + a_1 p_1 \cdots p_{r-1}.$$

This ends the proof. ☐

3.6. **Proof of Theorem (3.1.1).** We will use induction starting with the situation (3.4). Assume that the theorem is already verified for $\Gamma_{2,r}(\mathcal{I}_{2,r})$ and $\Gamma_{2,r}(\mathcal{J}_{2,r})$; we will check it in the next paragraphs for $(1, r)$. Clearly, it is enough to verify statement (1) of (3.1.1) only for $\mathcal{I}$ and (2) only for $\mathcal{J}$.

(1) Take $I \in \mathcal{I}$. If $s_0 = -\frac{\nu_{\ell,I}}{N_\ell}$ for some $\ell \geq 2$ is a pole of $Z(f_{1,r}, \omega_I; s)$, by (3.5.5)(ii) it is also a pole of $Z(f_{2,r}, \omega_{I'}; s)$ for some $I' \in \mathcal{I}_{2,r}$. Then the induction hypothesis says that $\exp(2\pi i s_0)$ is a root of $P(f_{2,r}; t) = \Delta_{a_2,p_2}(t^{p_3 \cdots p_r}) \cdot \ldots \cdot \Delta_{a_r,p_r}(t)$, and hence of $P(f_{1,r}; t)$. Suppose now that $s_0 = -\frac{\nu_{1,I}}{N_1}$ is a pole of $Z(f_{1,r}, \omega_I; s)$. By (3.1.2)(a) we can assume that $s_0 \neq -1$. Therefore $\mathcal{R}_1$ in (3.5.4) is the residue, hence $j_1 \neq a_1$ and $i_1 \neq p_1$; i.e. Case $A$ occurs. In other words, $j_1 \not\equiv 0 \pmod{a_1}$ and $i_1 \not\equiv 0 \pmod{p_1}$. But then, since $s_0 = -\frac{p_1 j_1 + a_1 i_1 - a_1 p_1 (\cdots)}{a_1 p_1 p_2 \cdots p_r}$, we have that $\exp(2\pi i s_0)$ is a root of $\Delta_{a_1,p_1}(t^{p_2 p_3 \cdots p_r})$.

(2) Take a root $\lambda$ of $P(f_{1,r};t)$. If it is a root of $\Delta_{a_2,p_2}(t^{p_3\cdots p_r})\cdot\ldots\cdot\Delta_{a_r,p_r}(t) = P(f_{2,r}t)$, the induction hypothesis yields a pole $s_0$ of $Z(f_{2,r},\omega_{I'};s)$ for some $I' \in \mathcal{J}_{2,r}$, hence by (3.5.5)(ii) also of $Z(f_{1,r},\omega_I;s)$ $(I \in \mathcal{J})$, such that $\exp(2\pi i s_0) = \lambda$.

Suppose on the other hand that $\lambda$ is a root of $\Delta_{a_1,p_1}(t^{p_2\cdots p_r})$. By (3.5.6), for some $I \in \mathcal{J}$ we get $\nu_{1,I}$ with $\exp(2\pi i(-\frac{\nu_{1,I}}{N_1})) = \lambda$. Since $\lambda \neq 1$, we get that $\nu_{1,I}/N_1 \neq 1$. Also, for such a $\lambda$ necessarily $j_1 \not\equiv 0 \pmod{a_1}$ and $i_1 \not\equiv 0 \pmod{p_1}$, and so $s_0 = -\frac{\nu_{1,I}}{N_1}$ is really a pole (of order 1) of $Z(f_{1,r},\omega_I;s)$, since the contribution $\mathcal{R}_1$ of (3.5.4) is really the residue of $s_0$.

Finally, assume that $\lambda = 1$, the eigenvalue of the monodromy on $H^0(F_0,\mathbb{C})$. Then take $\alpha = i_r = 1$ in the construction from the proof of (3.5.6). Then by the computation of that proof, we have $\nu_{1,I} = 1 + a_1p_1\ldots p_{r-1} < N_1$. Hence, $\nu_{\ell,I}/N_\ell < 1$ for all $\ell$, and $s = -1$ is indeed a pole of $Z(f_{1,r},\omega_I;s)$ for this choice of $I$.

This finishes the proof of Theorem (3.1.1).

3.7. **Example.** We describe the case of two Puiseux pairs more explicitly. A standard equation for $f$ is then

$$f = (y^{a_1} + x^{p_1})^{p_2} + x^u y^v$$

where $ua_1 + vp_1 = a_2$. The occurring $g, g_1, g_2$ in the forms $\omega_I, I = (j_1, i_1, i_2)$, can be chosen in the following easy way:

$$\omega_I = y^{j_1-1}x^{i_1-1}(y^{a_1} + x^{p_1})^{i_2-1}dx \wedge dy\,.$$

By the construction in (3.3) the sets $I$ in $\mathcal{J}$ are as follows. The number $i_2$ can take all values satisfying $0 < i_2 < p_2$. The couples $(j_1, i_1)$ can be either

$i_1 = p_1$ and $0 < j_1 \leq [\frac{a_2-1}{p_1}]$ (Case C),

$j_1 = a_1$ and $0 < i_1 \leq [\frac{a_2-1}{a_1}]$ but excluding here the multiples of $p_1$ (Case B),

or the values from Case A. In this last case it is in general not so clear how to describe really explicitly $(j_1, i_1)$ in terms of $a_1$, $p_1$ and $a_2$.

We mention a particular situation; suppose that $a_1 = p_1 + 1$. Then we can take $i_1 = \{\frac{j_2-1}{p_1}\}p_1 + 1$ and $j_1 = [\frac{j_2-1}{p_1}] + p_1 - \{\frac{j_2-1}{p_1}\}p_1$, where $j_2$ takes all values $0 < j_2 < a_2$.

3.8. **Remark.** Theorem (3.1.1) is also valid for the motivic or Hodge zeta functions associated to $f$ and the forms $\omega_I$, and, in the context of $p$-adic $f$, for its associated Igusa zeta functions.

Condition (2) for these zeta functions follows from the result for topological zeta functions, since the latter is a specialization of the former ones. (We refer to [18] for more details.) Concerning condition (1) we must be sure that in our case a pole of the motivic/Hodge/Igusa zeta function also yields a pole of the topological one, and this is true. The only non-obvious statement here is the 'if' part of the analogue of Lemma (3.5.4), and this is easily verified, for instance using [15].

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Rényi Institute of Mathematics, 1053 Budapest, Reáltanoda u. 13–15, Hungary
*E-mail address*: `nemethi@renyi.hu`
*URL*: `http://www.renyi.hu/~nemethi`

K.U.Leuven, Departement Wiskunde, Celestijnenlaan 200B, B–3001 Leuven, Belgium
*E-mail address*: `wim.veys@wis.kuleuven.be`
*URL*: `http://www.wis.kuleuven.be/algebra/veys.htm`