

## ZETA FUNCTIONS FOR ANALYTIC MAPPINGS, LOG-PRINCIPALIZATION OF IDEALS, AND NEWTON POLYHEDRA

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ABSTRACT. In this paper we provide a geometric description of the possible poles of the Igusa local zeta function  $Z_{\Phi}(s, \mathbf{f})$  associated to an analytic mapping  $\mathbf{f} = (f_1, \dots, f_l) : U(\subseteq K^n) \rightarrow K^l$ , and a locally constant function  $\Phi$ , with support in  $U$ , in terms of a log-principalization of the  $K[x]$ -ideal  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ . Typically our new method provides a much shorter list of possible poles compared with the previous methods. We determine the largest real part of the poles of the Igusa zeta function, and then as a corollary, we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of the subscheme given by  $\mathcal{I}_{\mathbf{f}}$ . We associate to an analytic mapping  $\mathbf{f} = (f_1, \dots, f_l)$  a Newton polyhedron  $\Gamma(\mathbf{f})$  and a new notion of non-degeneracy with respect to  $\Gamma(\mathbf{f})$ . The novelty of this notion resides in the fact that it depends on one Newton polyhedron, and Khovanskii's non-degeneracy notion depends on the Newton polyhedra of  $f_1, \dots, f_l$ . By constructing a log-principalization, we give an explicit list for the possible poles of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , in the case in which  $\mathbf{f}$  is non-degenerate with respect to  $\Gamma(\mathbf{f})$ .

### 1. INTRODUCTION

Let  $K$  be a  $p$ -adic field, i.e.  $[K : \mathbb{Q}_p] < \infty$ . Let  $R_K$  be the valuation ring of  $K$ ,  $P_K$  the maximal ideal of  $R_K$ , and  $\overline{K} = R_K/P_K$  the residue field of  $K$ . The cardinality of the residue field of  $K$  is denoted by  $q$ , thus  $\overline{K} = \mathbb{F}_q$ . For  $z \in K$ ,  $\text{ord}(z) \in \mathbb{Z} \cup \{+\infty\}$  denotes the valuation of  $z$ , and  $|z|_K = q^{-\text{ord}(z)}$  its absolute value. The absolute value  $|\cdot|_K$  can be extended to  $K^l$  by defining  $\|z\|_K = \max_{1 \leq i \leq l} |z_i|_K$ , for  $z = (z_1, \dots, z_l) \in K^l$ .

Let  $f_1, \dots, f_l$  be polynomials in  $K[x_1, \dots, x_n]$ , or, more generally,  $K$ -analytic functions on an open set  $U \subset K^n$ . We consider the mapping  $\mathbf{f} = (f_1, \dots, f_l) : K^n \rightarrow K^l$ , respectively,  $U \rightarrow K^l$ . Let  $\Phi : K^n \rightarrow \mathbb{C}$  be a Schwartz-Bruhat function (with support in  $U$  in the second case). The Igusa local zeta function associated to

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the above data is defined as

$$Z_{\Phi}(s, \mathbf{f}) = Z_{\Phi}(s, \mathbf{f}, K) = \int_{K^n} \Phi(x) \|\mathbf{f}(x)\|_K^s |dx|,$$

for  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 0$ , where  $|dx|$  is the Haar measure on  $K^n$  normalized in such a way that  $R_K^n$  has measure 1. We write  $Z(s, \mathbf{f})$ ,  $Z_0(s, \mathbf{f})$  and  $Z_W(s, \mathbf{f})$  when  $\Phi$  is the characteristic function of  $R_K^n$ ,  $P_K^n$ , and an open compact subset  $W$  of  $K^n$ , respectively.

The function  $Z_{\Phi}(s, \mathbf{f})$  admits a meromorphic continuation to the complex plane as a rational function of  $q^{-s}$ . Igusa established this result in the hypersurface case using Hironaka's resolution theorem [16, Theorem 8.2.1]. In the case  $l \geq 1$  the rationality of  $Z_{\Phi}(s, \mathbf{f})$  was established by Meuser in [24], however, as mentioned in the review MR 83g:12015 of [24], a trick by Serre allows to deduce the general case from the hypersurface case. Denef gave a completely different proof of the rationality of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , using  $p$ -adic cell decomposition [4]. The mentioned results do not give any information about the poles of  $Z_{\Phi}(s, \mathbf{f})$  in the case  $l > 1$ . In [37] the second author showed that a list of possible poles of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , can be computed from an embedded resolution of singularities of the divisor  $\cup_{i=1}^l f_i^{-1}(0)$  by using toroidal geometry. In the special case in which  $\mathbf{f}$  is a non-degenerate homogeneous polynomial mapping the possible poles of  $Z_{\Phi}(s, \mathbf{f})$  are given in [38].

In this paper we provide a geometric description of the possible poles of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , in terms of a log-principalization of the  $K[x]$ -ideal  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$  (see Theorem 2.4). At this point it is important to mention that the main result in [37] gives an algorithm to compute a list of possible poles of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , in terms of an embedded resolution of singularities of the divisor  $\cup_{i=1}^l f_i^{-1}(0)$ , while Theorem 2.4 gives a list of candidates to poles in terms of a log-principalization of the ideal  $\mathcal{I}_{\mathbf{f}}$ . Typically our new method provides a much shorter list of possible poles (see Example 2.5). It is important to mention that in the case  $l = 1$  the problem of determining the poles of the meromorphic continuation of  $Z_{\Phi}(s, \mathbf{f})$  in  $\operatorname{Re}(s) < 0$  has been studied extensively (see e.g. [3], [14], [28], [23], [32], [34]). The relevance of this problem is due to the existence of several conjectures relating the poles of  $Z_{\Phi}(s, \mathbf{f})$  with the structure of the singular locus of  $\mathbf{f}$ . In the case of polynomials in two variables, as a consequence of the works of Igusa, Strauss, Meuser and the first author, there is a complete solution of this problem [14], [27], [23], [33]. For general polynomials the problem of determination of the poles of  $Z_{\Phi}(s, \mathbf{f})$  is still open. There exists a generic class of polynomials named non-degenerate with respect to its Newton polyhedron for which it is possible to give a small set of candidates for the poles of  $Z_{\Phi}(s, \mathbf{f})$ . The poles of the local zeta functions attached to non-degenerate polynomials can be described in terms of Newton polyhedra. The case of two variables was studied by Lichtin and Meuser [21]. In [5], Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of  $Z_{\Phi}(s, \mathbf{f})$  in terms of the Newton polyhedron of  $f$ . This result was obtained by the second author, using an approach based on the  $p$ -adic stationary phase formula and Néron  $p$ -desingularization, for polynomials with coefficients in a non-archimedean local field of arbitrary characteristic [36], (see also [7], [29]).

In the case  $l = 1$ , among the conjectures relating the poles of Igusa's zeta function with topology and singularity theory, we mention here a conjecture of Igusa that proposes that the real parts of the poles of the Igusa zeta function of  $\mathbf{f}$  are roots

of the Bernstein polynomial of  $\mathbf{f}$  (see e.g. [3], [16], and references therein). It seems reasonable to believe that such relations between poles and singularity theory extend to the case  $l > 1$ . Indeed, recently it was proved that the above-mentioned conjecture of Igusa is valid in the case in which  $\mathcal{I}_{\mathbf{f}}$  is a monomial ideal [13].

In the case  $l = 1$ , the largest real part of the poles of the Igusa zeta function has been extensively studied both in the archimedean and non-archimedean cases [7], [21], [31], [36]. In the case  $l \geq 1$  we show that the largest real part  $-\lambda(\mathcal{I}_{\mathbf{f}})$  of the poles of the Igusa zeta function attached to  $\mathbf{f}$  can easily be determined from a log-principalization of the ideal  $\mathcal{I}_{\mathbf{f}}$  (see Theorem 2.7). As a consequence of this result we obtain an asymptotic estimation for the number of solutions of an arbitrary system of polynomial congruences in terms of the log-canonical threshold of a log-principalization (see Corollary 2.9, and the comments that follow). At this point we have to mention that in the case  $l = 1$  Loeser found lower and upper bounds for  $\lambda(\mathcal{I}_{\mathbf{f}})$  in terms of certain geometric invariants introduced by Teissier [22, Theorem 2.6 and Proposition 3.1.1], [30]. In this form he derived a geometric bound for the number of solutions of a polynomial congruence involving one polynomial.

If  $\mathbf{f}$  is a polynomial mapping with coefficients in a number field  $F$ , then for every maximal ideal  $P$  of the ring of algebraic integers of  $F$ , we can consider  $Z(s, \mathbf{f}, K)$ ,  $l \geq 1$ , where  $K$  is the completion of  $F$  with respect to  $P$ . We give an explicit formula for  $Z(s, \mathbf{f}, K)$ ,  $l \geq 1$ , that is valid for almost all  $P$  (see Theorem 2.10). The proof of this formula follows by adapting the argument given by Denef for the case  $l = 1$  [6].

One can also associate to a sheaf of ideals  $\mathcal{I}$  on a smooth algebraic variety (over a field of characteristic zero) a motivic zeta function (see Definition 2.16). By using a log-principalization of  $\mathcal{I}$  we give a similar explicit formula for it (see Theorem 2.17). The proof is a reasonably straightforward generalization of the one given by Denef and Loeser in [8]. By specializing to Euler characteristics one obtains the topological zeta function associated to  $\mathcal{I}$ .

We attach to an analytic mapping  $\mathbf{f} = (f_1, \dots, f_l)$  a Newton polyhedron  $\Gamma(\mathbf{f})$  and a new notion of non-degeneracy with respect to  $\Gamma(\mathbf{f})$ . The novelty of this notion resides in the fact that it depends on *one* Newton polyhedron, and Khovan-skii's non-degeneracy notion depends on the Newton polyhedra of  $f_1, \dots, f_l$  (see [18], [26]). By constructing a log-principalization, we give an explicit list for the possible poles of  $Z_{\Phi}(s, \mathbf{f})$ ,  $l \geq 1$ , in the case in which  $\mathbf{f}$  is non degenerate with respect to  $\Gamma(\mathbf{f})$  (see Theorem 3.11). This theorem provides a generalization to the case  $l \geq 1$  of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [5], [21], [36]. This result was originally established by Varchenko [31] for local zeta functions over  $\mathbb{R}$ . If  $\mathbf{f}$  is non-degenerate with respect to  $\Gamma(\mathbf{f})$ , then  $\lambda(\mathcal{I}_{\mathbf{f}})$  can be computed from  $\Gamma(\mathbf{f})$  in the classical way (see Corollary 3.12).

By using our notion of non-degeneracy and toroidal geometry we give an explicit formula for  $Z(s, \mathbf{f})$  and  $Z_0(s, \mathbf{f})$ ,  $l \geq 1$ . This formula generalizes one given by Denef and Hoornaert in the case  $l = 1$  [7, Theorem 4.2], and one given by the second author for the local zeta function of a monomial mapping [36, Theorem 6.1].

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## 2. THE IGUSA LOCAL ZETA FUNCTION OF A POLYNOMIAL MAPPING

**2.1. Log-principalization and poles of the Igusa local zeta function.** We state the two versions of log-principalization of ideals that we will use in this paper. The first is the ‘classical’ algebraic formulation, see for example [11], [12], [35]. The second is in the context of  $p$ -adic analytic functions. It follows from the results in [11], see 5.11 in that paper (noticing that ‘Property D’ there is valid in the  $p$ -adic analytic setting).

**Theorem 2.1** (Hironaka). *Let  $X_0$  be a smooth algebraic variety over a field of characteristic zero, and  $\mathcal{I}$  a sheaf of ideals on  $X_0$ . There exists a log-principalization of  $\mathcal{I}$ , that is a sequence*

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \dots \xleftarrow{\sigma_i} X_i \xleftarrow{\dots} \xleftarrow{\sigma_r} X_r = X$$

of blow-ups  $\sigma_i : X_{i-1} \leftarrow X_i$  in smooth centers  $C_{i-1} \subset X_{i-1}$  such that

- (1) the exceptional divisor  $E_i$  of the induced morphism  $\sigma^i = \sigma_1 \circ \dots \circ \sigma_i : X_i \rightarrow X_0$  has only simple normal crossings and  $C_i$  has simple normal crossings with  $E_i$ , and
- (2) the total transform  $(\sigma^r)^*(\mathcal{I})$  is the ideal of a simple normal crossings divisor  $E^\#$ . If the subscheme determined by  $\mathcal{I}$  has no components of codimension one, then  $E^\#$  is a natural combination of the irreducible components of the divisor  $E_r$ .

**Remark 2.2.** We use notations like  $(\sigma^r)^*(\mathcal{I})$  as in [35]. However, other authors use the notation  $\mathcal{IO}_X$  for the same object, for example in [11]. As many other authors we use the term ‘log-principalization’. The terms ‘principalization’ and ‘monomialization’ are also used for the same purpose by other authors.

**Theorem 2.3** ([11]). *Let  $K$  be a  $p$ -adic field and  $U$  an open submanifold of  $K^n$ . Let  $f_1, \dots, f_l$  be  $K$ -analytic functions on  $U$  such that the ideal  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$  is not trivial. Then there exists a log-principalization  $\sigma : X_K \rightarrow U$  of  $\mathcal{I}_{\mathbf{f}}$ , that is,*

- (1)  $X_K$  is an  $n$ -dimensional  $K$ -analytic manifold,  $\sigma$  is a proper  $K$ -analytic map which is a composition of a finite number of blow-ups in closed submanifolds, and which is an isomorphism outside of the common zero set  $Z_K$  of  $f_1, \dots, f_l$ ;
- (2)  $\sigma^{-1}(Z_K) = \cup_{i \in T} E_i$ , where the  $E_i$  are closed submanifolds of  $X_K$  of codimension one, each equipped with a pair of positive integers  $(N_i, v_i)$  satisfying the following. At every point  $b$  of  $X_K$  there exist local coordinates  $(y_1, \dots, y_n)$  on  $X_K$  around  $b$  such that, if  $E_1, \dots, E_p$  are the  $E_i$  containing  $b$ , we have on some neighborhood of  $b$  that  $E_i$  is given by  $y_i = 0$  for  $i = 1, \dots, p$ ,

$$\sigma^*(\mathcal{I}_{\mathbf{f}}) \text{ is generated by } \varepsilon(y) \prod_{i=1}^p y_i^{N_i},$$

and

$$\sigma^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) \left( \prod_{i=1}^p y_i^{v_i-1} \right) dy_1 \wedge \dots \wedge dy_n,$$

where  $\varepsilon(y), \eta(y)$  are units in the local ring of  $X_K$  at  $b$ .

The  $(N_i, v_i), i \in T$ , are called the numerical data of  $\sigma$ .

Let  $K$  be a  $p$ -adic field. Let  $f_1, \dots, f_l$  be polynomials over  $K$  or  $K$ -analytic functions on  $U \subset K^n$ . We set  $\mathcal{I}_{\mathbf{f}}$  to be the  $K$ -analytic ideal generated by the  $f_i$ ; we suppose it is not trivial. Let  $\Phi : K^n \rightarrow \mathbb{C}$  or  $U \rightarrow \mathbb{C}$  be a Schwartz-Bruhat function, that is, a locally constant function with compact support. We associate to  $\mathbf{f} = (f_1, \dots, f_l)$  and  $\Phi$  the Igusa zeta function  $Z_{\Phi}(s, \mathbf{f})$  as in the introduction. The following theorem yields a new proof of its meromorphic continuation, but especially it gives a list of its possible poles in terms of the numerical data of a log-principalization.

**Theorem 2.4.** *The local zeta function  $Z_{\Phi}(s, \mathbf{f})$  admits a meromorphic continuation to the complex plane as a rational function of  $q^{-s}$ . Furthermore, the poles have the form*

$$s = -\frac{v_i}{N_i} - \frac{2\pi\sqrt{-1}}{\log q} \frac{k}{N_i}, \quad k \in \mathbb{Z},$$

where the  $(N_i, v_i)$  are the numerical data of a log-principalization  $\sigma : X_K \rightarrow U$  of the ideal  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ .

*Proof.* We pick a log-principalization  $\sigma$  of  $\mathcal{I}_{\mathbf{f}}$  as in Theorem 2.3 and we use all notations that were introduced there.

At every point  $b \in X_K$  we can take a chart  $(V, \phi_V)$  with coordinates  $(y_1, \dots, y_n)$ , which may be shrunk later when necessary. Let  $g(y)$  be a generator of  $\sigma^*(\mathcal{I}_{\mathbf{f}}) = \sigma^*(f_1, \dots, f_l)$  in  $V$ . Then

$$g(y) = \varepsilon(y) \prod_{i=1}^p y_i^{N_i},$$

$$\sigma^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) \left( \prod_{i=1}^p y_i^{v_i-1} \right) dy_1 \wedge \dots \wedge dy_n,$$

where  $\varepsilon(y)$  and  $\eta(y)$  are units of the local ring of  $X_K$  at  $b$ . Furthermore, since  $\sigma^*(\mathcal{I}_{\mathbf{f}})$  is locally generated by  $g(y)$  we have

$$f_i^*(y) = g(y) \tilde{f}_i(y),$$

for  $i = 1, \dots, l$ ,  $y \in V$ , where each  $\tilde{f}_i(y)$  is an analytic function on  $V$ . And, since  $g(y) \in \sigma^*(\mathcal{I}_{\mathbf{f}})$ , we also have  $g(y) = \sum_{i=1}^l a_i(y) f_i^*(y)$ , with  $a_i(y)$  an analytic function on  $V$  for each  $i$ ; therefore

$$1 = \sum_{i=1}^l a_i(y) \tilde{f}_i(y), \text{ for } y \in V.$$

Then there exists at least one index  $i_0$  such that  $\tilde{f}_{i_0}(b) \neq 0$ , hence we may assume that  $\tilde{f}_{i_0}(y) \neq 0$  on  $V$  and that

$$\|(f_1^*(y), \dots, f_l^*(y))\|_K^s = \left\| \left( (\tilde{f}_i(y))_{i \notin H}, (\tilde{f}_i(b))_{i \in H} \right) \right\|_K^s |g(y)|_K^s,$$

for  $y \in V$ . Here  $H \subseteq \{1, \dots, n\}$  such that  $\tilde{f}_i(b) \neq 0 \Leftrightarrow i \in H$ . We may further suppose that

$$\left\| \left( (\tilde{f}_i(y))_{i \notin H}, (\tilde{f}_i(b))_{i \in H} \right) \right\|_K^s = \left\| (\tilde{f}_i(b))_{i \in H} \right\|_K^s$$

on  $V$ . Since  $\sigma$  is proper,  $\sigma^{-1}(\text{supp}(\Phi))$  is compact open in  $X_K$ , hence we can express it as a finite disjoint union of compact open sets  $B_\alpha$  such that each  $B_\alpha$  is contained in some  $V$  above. Since  $\Phi$  is locally constant we may assume (after subdividing  $B_\alpha$ ) that  $(\Phi \circ \sigma)|_{B_\alpha} = (\Phi \circ \sigma)(b)$ ,  $|\varepsilon|_K|_{B_\alpha} = |\varepsilon(b)|_K$ ,  $|\eta|_K|_{B_\alpha} = |\eta(b)|_K$ , and  $\phi_V(B_\alpha) = c + \pi^{e_0} R_K^n$ .

Denote by  $D_K = (\text{div}(\sigma^*(\mathcal{I}_f)))_K$ . Since  $\sigma : X_K \setminus D_K \rightarrow U \setminus \sigma(D_K)$  is bianalytic, and  $D_K$  has measure zero, we have

$$\begin{aligned} Z_\Phi(s, \mathbf{f}) &= \int_{U \setminus \sigma(D_K)} \Phi(x) \|\mathbf{f}(x)\|_K^s |dx| \\ &= \sum_\alpha (\Phi \circ \sigma)(b) |\varepsilon(b)|_K^s |\eta(b)|_K \left\| \left( \tilde{f}_i(b) \right)_{i \in H} \right\|_K^s \int_{c + \pi^{e_0} R_K^n} \prod_{1 \leq i \leq p} |y_i|^{N_i s + v_i - 1} |dy|. \end{aligned}$$

The conclusion is now obtained by computing the integral in the previous expression like in the case  $l = 1$  (see [16, Lemma 8.2.1]).  $\square$

**Example 2.5.** Let  $K$  be a  $p$ -adic field, and let  $f_1(x, y) = y^a - x^b$ ,  $f_2(x, y) = x^a - y^b$ , with  $a < b$ , and for  $j = 3, \dots, M$ ,  $M \geq 3$ ,  $f_j(x, y) = x^{n_j} y^{m_j} h_j(x, y)$ , with  $n_j, m_j \geq a$ , and  $h_j(x, y) \in K[x, y]$ . Set  $\mathbf{f} = (f_1, f_2, f_3, \dots, f_M)$ , and  $I_{\mathbf{f}} = (f_1, f_2, f_3, \dots, f_M)$ . Let  $\Phi$  be a Schwartz-Bruhat function whose support is contained in a sufficiently small neighborhood of the origin. A log-principalization of the ideal  $I_{\mathbf{f}}$  (over a neighborhood of the origin) is obtained by blowing-up the origin of  $K^2$ . There is only one exceptional curve  $E = \mathbb{P}^1(K)$  whose numerical datum is  $(a, 2)$ , and therefore the possible poles of  $Z_\Phi(s, \mathbf{f})$  have real part  $\frac{-2}{a}$ . In [37] an algorithm for computing a list of candidates for the poles of  $Z_\Phi(s, \mathbf{f})$  in terms of the numerical data of an embedded resolution of the divisor  $\cup_{j=1}^M f_j^{-1}(0)$  was given. Since the  $f_j(x, y)$  are arbitrary polynomials for  $3 \leq j \leq M$ , the mentioned algorithm gives in general a very long list of possible poles.

**2.2. The largest real part of the poles of the Igusa zeta function.** Let  $U$  be a compact open subset of  $K^n$  and let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be an analytic mapping. Recall that  $Z_U(s, \mathbf{f}) = \int_U \|\mathbf{f}(x)\|_K^s |dx|$ . The following lemma is known by the experts, however we did not find a suitable reference for it; for the sake of completeness we include its proof here.

**Lemma 2.6.** (1)  $Z_U(s, \mathbf{f})$  has no pole in  $s$ , i.e.  $Z_U(s, \mathbf{f})$  is a Laurent polynomial in  $q^{-s}$  if and only if there is no  $x \in U$  such that  $f_1(x) = \dots = f_l(x) = 0$ .  
(2) If  $0 \in U$  and  $\mathbf{f}(0) = 0$ , i.e.  $f_1(0) = \dots = f_l(0) = 0$ , then  $Z_U(s, \mathbf{f})$  has at least one pole in  $s$ .

*Proof.* (1) We first note that rationality of  $Z_U(s, \mathbf{f})$  implies the equivalence of the conditions “ $Z_U(s, \mathbf{f})$  has no pole in  $s$ ” and “ $Z_U(s, \mathbf{f})$  is a Laurent polynomial in  $q^{-s}$ .”

( $\Leftarrow$ ) Since  $\mathbf{f} : U \rightarrow K^l$  is continuous, also  $\|\mathbf{f}\|_K : U \rightarrow q^{\mathbb{Z}} \cup \{0\}$  is continuous. If 0 does not belong to the image of  $\|\mathbf{f}\|_K$ , then there are only finitely many values in the image because  $U$  is compact. So  $\int_U \|\mathbf{f}(x)\|_K^s |dx|$  is a Laurent polynomial in  $q^{-s}$ .

( $\Rightarrow$ ) If  $x_0 \in U$  with  $f_1(x_0) = \dots = f_l(x_0) = 0$ , by using the continuity of  $\|\mathbf{f}\|_K$ , there exist infinitely many  $i$  such that there exists  $x_i \in U$  with  $\|\mathbf{f}(x_i)\|_K = q^{-i}$ .

Since  $U$  is open we have for all those  $i$  that the Haar measure of the set

$$\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-i}\}$$

is positive. Therefore

$$Z_U(s, \mathbf{f}) = \sum_j \text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-j}\}) q^{-sj}$$

is not a Laurent polynomial in  $q^{-s}$ .

(2) The second part follows directly from the first one.  $\square$

**Theorem 2.7.** *Let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be an analytic mapping defined on a compact open neighborhood of the origin  $U$  such that  $\mathbf{f}(0) = 0$ . We take a log-principalization  $\sigma : X_K \rightarrow U$  as in Theorem 2.3 with numerical data  $(N_i, v_i)$ ,  $i \in T$ . Let  $\lambda := \lambda(\mathcal{I}_{\mathbf{f}}) = \min_i \frac{v_i}{N_i}$ . Then  $-\lambda(\mathcal{I}_{\mathbf{f}})$  is the real part of a pole of  $Z_U(s, \mathbf{f})$ . In particular,  $\lambda(\mathcal{I}_{\mathbf{f}})$  depends only on  $\mathcal{I}_{\mathbf{f}}$ .*

*Proof.* The proof will be achieved by establishing that  $q^\lambda$  is the radius of convergence  $\mathbf{R}$  of  $Z_U(s, \mathbf{f})$  considered as a function in  $q^{-s}$ . Certainly  $\mathbf{R} \geq q^\lambda$ , since (by Theorem 2.4) the candidate poles closest to the origin have modulus  $q^\lambda$ . We shall show that  $\mathbf{R} \leq q^\lambda$  by proving a lower bound for the coefficients of  $Z_U(q^{-s}, \mathbf{f})$ , considered as power series in  $q^{-s}$ :

$$Z_U(q^{-s}, \mathbf{f}) = \sum_j \text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-j}\}) q^{-sj}.$$

Take a generic point  $b$  on a component  $E_r$  with  $\frac{v_r}{N_r} = \lambda$ , and a small enough chart  $B \subset X_K$  around  $b$  with coordinates  $(y_1, \dots, y_n)$  such that

$$\sigma^*(\mathcal{I}_{\mathbf{f}}) \text{ is generated by } \varepsilon(y) y_1^{N_r},$$

and

$$\sigma^*(dx_1 \wedge \dots \wedge dx_n) = \eta(y) y_1^{v_r-1} dy_1 \wedge \dots \wedge dy_n,$$

on  $B$ , where  $|\varepsilon|_K$  and  $|\eta|_K$  are constant (and nonzero) on  $B$ . After an eventual  $K$ -analytic coordinate change, we may assume furthermore that  $B = R_K^n$ .

**Claim.** *For  $j$  big enough and divisible by  $N_r$  we have*

$$\text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-j}\}) \geq Cq^{-j\lambda},$$

where  $C$  is a positive constant.

By the above claim we have

$$\limsup_{i \rightarrow \infty} [\text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-i}\})]^{1/i} \geq q^{-\lambda}$$

and hence

$$\mathbf{R} = \frac{1}{\limsup_{i \rightarrow \infty} [\text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-i}\})]^{1/i}} \leq q^\lambda.$$

Therefore, since  $Z_U(q^{-s}, \mathbf{f})$  is a rational function of  $q^{-s}$ , we conclude that  $uq^\lambda$  is a pole of  $Z_U(q^{-s}, \mathbf{f})$ , for some complex  $N_r$ -th root of the unity  $u$ .

**Proof of the claim.** By the  $p$ -adic change of variables formula [16, Proposition 7.4.1] we have ( $B \subset \sigma^{-1}(U)$ ):

$$(2.1) \quad \text{vol}(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-j}\}) \geq \text{vol}(\{y \in B \mid \|\mathbf{f} \circ \sigma(y)\|_K = q^{-j}\}) \cdot |(Jac \sigma)(y)|_K,$$

where  $Jac \sigma$  is the Jacobian determinant of  $\sigma$ . With the same reasoning as in the proof of Theorem 2.4 we have that  $\|\mathbf{f} \circ \sigma(y)\|_K = C_1 |\varepsilon|_K |y_1|_K^{N_r}$  on  $B$ , where  $C_1$  is a positive constant. So on  $B$  we have  $\|\mathbf{f} \circ \sigma(y)\|_K = q^{-j}$  if and only if  $|y_1|_K = C_2 q^{-j/N_r}$ , where  $C_2$  is a positive constant. Hence

$$(2.2) \quad vol(\{y \in B \mid \|\mathbf{f} \circ \sigma(y)\|_K = q^{-j}\}) = (1 - q^{-1}) C_2 q^{-j/N_r}.$$

Note that on this subset of  $B$  we have

$$(2.3) \quad |(Jac \sigma)(y)|_K = |\eta|_K |y_1|_K^{v_r-1} = |\eta|_K C_2^{v_r-1} q^{-j(v_r-1)/N_r}.$$

Combining (2.1), (2.2) and (2.3) yields

$$vol(\{x \in U \mid \|\mathbf{f}(x)\|_K = q^{-j}\}) \geq C q^{-\lambda j},$$

for some positive constant  $C$ . □

**Remark 2.8.** (1) In [15] Igusa showed in the case  $l = 1$  that  $-\lambda(\mathcal{I}_{\mathbf{f}})$  is a pole of  $Z_U(s, \mathbf{f})$  for a suitable compact open set  $U$  containing the origin. The argument uses Langlands' description of residues in terms of principal value integrals [20]. Furthermore, this argument is valid for archimedean and non-archimedean local zeta functions (see also [2, Théorème 5, part 3a, page 186], [31]).

(2) We note that  $\lambda(\mathcal{I}_{\mathbf{f}}) \geq lct(\mathcal{I}_{\mathbf{f}})$ , where  $lct(\mathcal{I}_{\mathbf{f}})$  is the 'log-canonical threshold' of  $\mathcal{I}_{\mathbf{f}}$ . This well-known important invariant (see e.g. [19], [25]) is defined analogously as  $\lambda(\mathcal{I}_{\mathbf{f}})$  but in a geometric setting, i.e. working over an algebraic closure of  $K$ . In order to obtain a log-principalization in this context maybe more exceptional components are needed, and then the inequality above could be strict.

2.2.1. *Number of solutions of polynomial congruences.* Suppose that  $f_i(x)$ ,  $i = 1, \dots, l$ , are polynomials with coefficients in  $R_K$ . Let  $N_j(\mathbf{f})$  be the number of solutions of  $f_i(x) \equiv 0 \pmod{P_K^j}$ ,  $i = 1, \dots, l$ , in  $(R_K/P_K^j)^n$ , and let  $P(t, \mathbf{f})$  be the series  $\sum_{j=0}^{\infty} N_j(\mathbf{f})(q^{-n}t)^j$ . The Poincaré series  $P(t, \mathbf{f})$  is related to  $Z(s, \mathbf{f})$  by the formula  $P(t, \mathbf{f}) = \frac{1-tZ(s, \mathbf{f})}{1-t}$ ,  $t = q^{-s}$ , (cf. [24, Theorem 2]). In the proof of the previous theorem was established that  $q^\lambda$  is the radius of convergence  $R$  of  $Z(s, \mathbf{f})$  considered as a function in  $q^{-s}$ . By using this fact, and the above-mentioned relation between  $P(t, \mathbf{f})$  and  $Z(s, \mathbf{f})$ , we obtain the following corollary.

**Corollary 2.9.** *With the above notation,*

$$\limsup_{j \rightarrow \infty} [N_j(\mathbf{f})q^{-nj}]^{\frac{1}{j}} = q^{-\lambda(\mathcal{I}_{\mathbf{f}})},$$

where  $\lambda(\mathcal{I}_{\mathbf{f}}) = \min \left\{ \frac{v_i}{N_i} \right\}$ , where  $(N_i, v_i)$  runs through the numerical data of a log-principalization  $\sigma : X_K \rightarrow R_K^n$  of the ideal  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$ .

Let  $d$  be the maximal order of the poles of  $P(t, \mathbf{f})$  with modulus  $q^{\lambda(\mathcal{I}_{\mathbf{f}})}$ . As a consequence of the above corollary and of the rationality of  $P(t, \mathbf{f})$  we have that  $N_j(\mathbf{f}) \leq C j^{d-1} q^{(n-\lambda(\mathcal{I}_{\mathbf{f}}))j}$  for  $j$  big enough, where  $C$  is a positive constant. And by Remark 2.8 (2), we have then that  $N_j(\mathbf{f}) \leq C j^{d-1} q^{(n-lct(\mathcal{I}_{\mathbf{f}}))j}$  for  $j$  big enough.

**2.3. Denef's explicit formula.** For polynomials  $f_1, \dots, f_l$  over a number field  $F$ , we can consider local zeta functions  $Z_W(s, \mathbf{f}, K)$  for all (non-archimedean) completions  $K$  of  $F$ . When  $l = 1$ , Denef presented in [6, Theorem 3.1] an explicit formula, which is valid simultaneously for almost all these zeta functions. His arguments extend to the several polynomials case, by replacing resolution by log-principalization (as in Theorem 2.1).

**Theorem 2.10.** *Let  $F$  be a number field and  $f_i(x) \in F[x_1, \dots, x_n]$  for  $i = 1, \dots, l$ . Let  $\sigma : X \rightarrow \mathbb{A}^n$  be a log-principalization of  $\mathcal{I}_{\mathbf{f}} = (f_1, \dots, f_l)$  over  $F$  as in Theorem 2.1. Denote  $\text{div}(\sigma^*(\mathcal{I}_{\mathbf{f}})) = \sum_{i \in T} N_i E_i$ , and  $\text{div}(\sigma^*(dx_1 \wedge \dots \wedge dx_n)) = \sum_{i \in T} (v_i - 1) E_i$ , where  $E_i$ ,  $i \in T$ , are the irreducible components of the simple normal crossings divisor given by the principal ideal  $\sigma^*(\mathcal{I}_{\mathbf{f}})$ . For every maximal ideal  $P$  of the ring of integers of  $F$ , we consider the completion  $K$  of  $F$  with respect to  $P$ . Denote the valuation ring and the residue field of  $K$  by  $R$  and  $\overline{K} = \mathbb{F}_q$  respectively. Then for almost all completions  $K$  (i.e. for all except a finite number) we have*

$$Z_W(s, \mathbf{f}, K) = q^{-n} \sum_{I \subseteq T} c_I \prod_{i \in I} \frac{(q-1) q^{-N_i s - v_i}}{1 - q^{-N_i s - v_i}},$$

where  $W \subset R^n$  is a union of cosets mod  $(P)^n$ , and

$$c_I = \text{card} \{a \in \overline{X}(\overline{K}) \mid a \in \overline{E}_i(\overline{K}) \Leftrightarrow i \in I; \text{ and } \overline{\sigma}(a) \in \overline{W}\}.$$

Here  $\overline{\cdot}$  denotes the reduction mod  $P$ , for which we refer to [6, Sect. 2].

**Example 2.11.** Take  $f_1, f_2, f_3, \dots, f_M$  as in Example 2.5 as being defined over a number field  $F$ . Then the formula of Theorem 2.10 for  $W = (P)^2$  yields

$$Z_0(s, \mathbf{f}, K) = q^{-2} (q+1) \frac{(q-1) q^{-as-2}}{1 - q^{-as-2}} = \frac{(1 - q^{-2}) q^{-as-2}}{1 - q^{-as-2}}.$$

**Example 2.12.** Let  $K = \mathbb{Q}_p$ ,  $f_1(x, y) = x$ ,  $f_2(x, y) = x + p_0 y$ , where  $p_0$  is a fixed prime number, and let  $\mathbf{f} = (f_1, f_2)$ . A direct calculation shows that

$$Z(s, \mathbf{f}, K) = \begin{cases} \frac{1-p^{-2}}{1-p^{-2-s}}, & p \neq p_0, \\ \frac{(1-p^{-1})(1+p^{-1-s})}{1-p^{-2-s}}, & p = p_0. \end{cases}$$

A log-principalization for the ideal  $\mathcal{I}_{\mathbf{f}}$  is attained by blowing-up the origin. One easily verifies that the expression for  $p \neq p_0$  is the one given by Theorem 2.10.

As a consequence of Theorem 2.4 (or [4], [24])  $Z_W(s, \mathbf{f})$  can be written as

$$Z_W(s, \mathbf{f}) = \frac{P(T)}{Q(T)},$$

where  $P(T)$  and  $Q(T)$  are polynomials in  $T = q^{-s}$  with rational coefficients. We define  $\text{deg } Z_W(s, \mathbf{f}) = \text{deg } P(T) - \text{deg } Q(T)$ , where  $\text{deg}$  means 'degree'.

**Corollary 2.13.** *Let  $f_i(x) \in F[x_1, \dots, x_n]$  for  $i = 1, \dots, l$ . For almost all completions  $K$  of  $F$  we have  $\text{deg } Z(s, \mathbf{f}, K) \leq 0$  and  $\text{deg } Z_0(s, \mathbf{f}, K) = 0$ . Moreover if all  $f_i$  are homogeneous of degree  $d$ , then  $\text{deg } Z(s, \mathbf{f}, K) = -d$ .*

The proof follows from the explicit formula (Theorem 2.10) by analogous arguments as in [6] (or [16]) where the case  $l = 1$  is treated. We should mention that by using model-theoretic arguments Denef already showed the above result (see

[6, Theorem 5.2, and Example 5.4]). So in this paper we give a geometric proof of this fact.

Note that for the case  $p = p_0$  in Example 2.12 it is not true that  $\deg Z(s, \mathbf{f}, \mathbb{Q}_p) = -1$ , though  $f_1, f_2$  are homogeneous of degree 1.

**Example 2.14.** Let  $\mathbf{f} = (f_1, f_2) = (x^3 - xy, y)$ . One easily constructs a log-principalization of the ideal  $\mathcal{I}_{\mathbf{f}} = (x^3 - xy, y)$  as a composition of three blow-ups. The numerical data of the three exceptional components in  $\sigma^{-1}(\text{supp } \mathcal{I}_{\mathbf{f}}) = \sigma^{-1}(0)$  are  $(1, 2)$ ,  $(2, 3)$ ,  $(3, 4)$  respectively. So Theorem 2.4 yields  $-2, -3/2, -4/3$  as possible (real parts of) candidate poles of  $Z(s, \mathbf{f})$ . However, in the formula of Theorem 2.10 the first two candidate poles cancel:

$$\begin{aligned} Z(s, \mathbf{f}) &= q^{-2} \left\{ (q^2 - 1) + q \frac{(q-1)q^{-2-s}}{1-q^{-2-s}} + (q-1) \frac{(q-1)q^{-3-2s}}{1-q^{-3-2s}} + q \frac{(q-1)q^{-4-3s}}{1-q^{-4-3s}} \right. \\ &\quad \left. + \frac{(q-1)^2 q^{-5-3s}}{(1-q^{-2-s})(1-q^{-3-2s})} + \frac{(q-1)^2 q^{-7-5s}}{(1-q^{-3-2s})(1-q^{-4-3s})} \right\} \\ &= q^{-2} \frac{q-1}{1-q^{-4-3s}} (q+1 + q^{-1-s} + q^{-2-2s}). \end{aligned}$$

We shall present an alternative formula to compute this example in Section 4, where only one candidate pole will appear.

**Example 2.15.** Let  $\mathbf{f} = (f_1, f_2) = (y^2 - x^3, y^2 - z^2)$ . We shall compute  $Z_0(s, \mathbf{f})$  by means of a log-principalization of  $\mathcal{I}_{\mathbf{f}} = (y^2 - x^3, y^2 - z^2)$ . Note that the support of  $\mathcal{I}_{\mathbf{f}}$  has two 1-dimensional components  $C$  and  $C'$  with a singularity at the origin of  $K^3$ .

We first blow up the origin yielding the exceptional surface  $E_1 (\cong \mathbb{P}^2)$  with  $(N_1, v_1) = (2, 3)$ . The strict transform of  $C$  and  $C'$  and  $E_1$  have one common point. Next we blow up this point obtaining the new exceptional surface  $E_2 (\cong \mathbb{P}^2)$  with  $(N_2, v_2) = (3, 5)$ . At this stage (the strict transforms of)  $C$  and  $C'$  are disjoint and both meet  $E_2$  in one point of the intersection of  $E_2$  with (the strict transform of)  $E_1$ . Now we blow up the curve  $E_1 \cap E_2$ ; the new exceptional component  $E_3$  is a ruled surface over that curve and  $(N_3, v_3) = (6, 8)$ . We have that  $E_3 \cap E_1$  and  $E_3 \cap E_2$  are disjoint sections of  $E_3$ , and  $C$  and  $C'$  intersect  $E_3$  transversely outside  $E_3 \cap E_1$  and  $E_3 \cap E_2$ . Finally we blow up  $C$  and  $C'$ , yielding the last two exceptional surfaces  $E_4$  and  $E'_4$  with numerical data  $(1, 2)$ . The formula of Theorem 2.10 yields

$$\begin{aligned} Z_0(s, \mathbf{f}) &= q^{-3} \left( (q^2 + q) \frac{(q-1)q^{-3-2s}}{1-q^{-3-2s}} + q^2 \frac{(q-1)q^{-5-3s}}{1-q^{-5-3s}} \right. \\ &\quad \left. + (q^2 - 3) \frac{(q-1)q^{-8-6s}}{1-q^{-8-6s}} + (q+1) \frac{(q-1)^2 q^{-11-8s}}{(1-q^{-3-2s})(1-q^{-8-6s})} \right. \\ &\quad \left. + (q+1) \frac{(q-1)^2 q^{-13-9s}}{(1-q^{-5-3s})(1-q^{-8-6s})} + 2(q+1) \frac{(q-1)^2 q^{-10-7s}}{(1-q^{-2-s})(1-q^{-8-6s})} \right) \\ &= q^{-3} (q-1) \frac{N(q^{-s})}{(1-q^{-2-s})(1-q^{-8-6s})}, \end{aligned}$$

where

$$\begin{aligned} N(q^{-s}) &= (q^2 - q - 1)q^{-10-7s} + (q^2 + q - 1)q^{-8-6s} - (q+1)q^{-7-5s} \\ &\quad + q^{-4-4s} - q^{-4-3s} + (q+1)q^{-2-2s}. \end{aligned}$$

Note that the candidate poles  $-3/2$  and  $-5/3$  cancel.

**2.4. Motivic and topological zeta functions.** The analogue of the original explicit formula of Denef plays an important role in the study of the motivic zeta function associated to one regular function [8]. One can associate more generally a motivic zeta function to any sheaf of ideals on a smooth variety, and obtain a similar formula for it in terms of a log-principalization using the argument of [8]. We just formulate the more general definition and formula, referring to e.g. [9], [32] for the notion of jets and Grothendieck ring.

**Definition 2.16.** Let  $Y$  be a smooth algebraic variety of dimension  $n$  over a field  $F$  of characteristic zero, and  $\mathcal{I}$  a sheaf of ideals on  $Y$ . Let  $W$  be a subvariety of  $Y$ . Denote for  $i \in \mathbb{N}$  by  $\mathfrak{X}_{i,W}$  the variety of  $i$ -jets  $\gamma$  on  $Y$  with origin in  $W$  for which  $\text{ord}_t(\gamma^*\mathcal{I}) = i$ . The *motivic zeta function* associated to  $\mathcal{I}$  (and  $W$ ) is the formal power series

$$Z_W(\mathcal{I}, T) = \sum_{i \geq 0} [\mathfrak{X}_{i,W}] (\mathbb{L}^{-n}T)^i,$$

where  $[\cdot]$  denotes the class of a variety in the Grothendieck ring of algebraic varieties over  $F$ , and  $\mathbb{L} = [\mathbb{A}^1]$ .

**Theorem 2.17.** Let  $\sigma : X \rightarrow Y$  be a log-principalization of  $\mathcal{I}$ . With the analogous notation  $E_i, N_i, v_i, (i \in T)$  as before, and also  $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{k \notin I} E_k)$  for  $I \subset T$ , we have

$$Z_W(\mathcal{I}, T) = \sum_{I \subset T} [E_I^\circ \cap \sigma^{-1}W] \prod_{i \in I} \frac{(\mathbb{L} - 1) T^{N_i}}{\mathbb{L}^{v_i} - T^{N_i}}.$$

In particular  $Z_W(\mathcal{I}, T)$  is rational in  $T$ .

Specializing to topological Euler characteristics, denoted by  $\chi(\cdot)$ , as in [8, (2.3)] or [32, (6.6)] we obtain the expression

$$Z_{top,W}(\mathcal{I}, s) := \sum_{I \subset T} \chi(E_I^\circ \cap \sigma^{-1}W) \prod_{i \in I} \frac{1}{v_i + N_i s} \in \mathbb{Q}(s),$$

which is then independent of the chosen log-principalization. (When the base field is not the complex numbers, we consider  $\chi(\cdot)$  in étale  $\mathbb{Q}$ -cohomology as in [8].) It can be taken as a definition for the *topological zeta function* associated to  $\mathcal{I}$  (and  $W$ ), generalizing the original one of Denef and Loeser associated to one polynomial [10].

### 3. NEWTON POLYHEDRA AND NON-DEGENERACY CONDITIONS

**3.1. Newton polyhedra.** We set  $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ .

Let  $G$  be a nonempty subset of  $\mathbb{N}^n$ . The *Newton polyhedron*  $\Gamma = \Gamma(G)$  associated to  $G$  is the convex hull in  $\mathbb{R}_+^n$  of the set  $\cup_{m \in G} (m + \mathbb{R}_+^n)$ . For instance classically one associates a *Newton polyhedron (at the origin)* to  $g(x) = \sum_m c_m x^m$  ( $x = (x_1, \dots, x_n)$ ,  $g(0) = 0$ ), being a nonconstant polynomial function over  $K$  or  $K$ -analytic function in a neighborhood of the origin, where  $G = \text{supp}(g) := \{m \in \mathbb{N}^n \mid c_m \neq 0\}$ . Further we will associate more generally a Newton polyhedron to an analytic mapping.

We fix a Newton polyhedron  $\Gamma$  as above. We first collect some notions and results about Newton polyhedra that will be used in the next sections. Let  $\langle \cdot, \cdot \rangle$

denote the usual inner product of  $\mathbb{R}^n$ , and identify the dual space of  $\mathbb{R}^n$  with  $\mathbb{R}^n$  itself by means of it.

For  $a \in \mathbb{R}_+^n$ , we define

$$d(a, \Gamma) = d(a) = \min_{x \in \Gamma} \langle a, x \rangle,$$

and the first meet locus  $F(a)$  of  $a$  as

$$F(a) := \{x \in \Gamma \mid \langle a, x \rangle = d(a)\}.$$

The first meet locus is a face of  $\Gamma$ . Moreover, if  $a \neq 0$ ,  $F(a)$  is a proper face of  $\Gamma$ .

We define an equivalence relation in  $\mathbb{R}_+^n$  by taking  $a \sim a' \Leftrightarrow F(a) = F(a')$ . The equivalence classes of  $\sim$  are sets of the form

$$\Delta_\tau = \{a \in \mathbb{R}_+^n \mid F(a) = \tau\},$$

where  $\tau$  is a face of  $\Gamma$ .

We recall that the cone strictly spanned by the vectors  $a_1, \dots, a_r \in \mathbb{R}_+^n \setminus \{0\}$  is the set  $\Delta = \{\lambda_1 a_1 + \dots + \lambda_r a_r \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$ . If  $a_1, \dots, a_r$  are linearly independent over  $\mathbb{R}$ ,  $\Delta$  is called a *simplicial cone*. If  $a_1, \dots, a_r \in \mathbb{Z}^n$ , we say  $\Delta$  is a *rational cone*. If  $\{a_1, \dots, a_r\}$  is a subset of a basis of the  $\mathbb{Z}$ -module  $\mathbb{Z}^n$ , we call  $\Delta$  a *simple cone*.

A precise description of the geometry of the equivalence classes modulo  $\sim$  is as follows. Each *facet* (i.e. a face of codimension one)  $\gamma$  of  $\Gamma$  has a unique vector  $a(\gamma) = (a_{\gamma,1}, \dots, a_{\gamma,n}) \in \mathbb{N}^n \setminus \{0\}$ , whose nonzero coordinates are relatively prime, which is perpendicular to  $\gamma$ . We denote by  $\mathfrak{D}(\Gamma)$  the set of such vectors. The equivalence classes are rational cones of the form

$$\Delta_\tau = \left\{ \sum_{i=1}^r \lambda_i a(\gamma_i) \mid \lambda_i \in \mathbb{R}_+, \lambda_i > 0 \right\},$$

where  $\tau$  runs through the set of faces of  $\Gamma$ , and  $\gamma_i$ ,  $i = 1, \dots, r$  are the facets containing  $\tau$ . We note that  $\Delta_\tau = \{0\}$  if and only if  $\tau = \Gamma$ . The family  $\{\Delta_\tau\}_\tau$ , with  $\tau$  running over the proper faces of  $\Gamma$ , is a partition of  $\mathbb{R}_+^n \setminus \{0\}$ ; we call this partition a *polyhedral subdivision of  $\mathbb{R}_+^n$  subordinated to  $\Gamma$* . We call  $\{\overline{\Delta}_\tau\}_\tau$ , the family formed by the topological closures of the  $\Delta_\tau$ , a *fan subordinated to  $\Gamma$* .

Each cone  $\Delta_\tau$  can be partitioned into a finite number of simplicial cones  $\Delta_{\tau,i}$ . In addition, the subdivision can be chosen such that each  $\Delta_{\tau,i}$  is spanned by part of  $\mathfrak{D}(\Gamma)$ . Thus from the above considerations we have the following partition of  $\mathbb{R}_+^n \setminus \{0\}$ :

$$(3.1) \quad \mathbb{R}_+^n \setminus \{0\} = \bigcup_{\tau} \left( \bigcup_{i=1}^{l_\tau} \Delta_{\tau,i} \right),$$

where  $\tau$  runs over the proper faces of  $\Gamma$ , and each  $\Delta_{\tau,i}$  is a simplicial cone contained in  $\Delta_\tau$ . We will say that  $\{\Delta_{\tau,i}\}$  is a *simplicial polyhedral subdivision of  $\mathbb{R}_+^n$  subordinated to  $\Gamma$* ; and that  $\{\overline{\Delta}_{\tau,i}\}$  is a *simplicial fan subordinated to  $\Gamma$* .

By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones. In this way we obtain a *simple polyhedral subdivision of  $\mathbb{R}_+^n$  subordinated to  $\Gamma$* ; and a *simple fan subordinated to  $\Gamma$*  (see e.g. [17]).

**3.2. The Newton polyhedron associated to an analytic mapping.** Let  $\mathbf{f} = (f_1, \dots, f_l)$ ,  $\mathbf{f}(0) = 0$ , be a nonconstant polynomial mapping, or more generally, an analytic mapping defined on a neighborhood  $U \subseteq K^n$  of the origin. In this paper we associate to  $\mathbf{f}$  a Newton polyhedron  $\Gamma(\mathbf{f}) := \Gamma(\cup_{i=1}^l \text{supp}(f_i))$ , and a non-degeneracy condition to  $\mathbf{f}$  and  $\Gamma(\mathbf{f})$ .

If  $f_i(x) = \sum_m c_{m,i} x^m$ , and  $\tau$  is a face of  $\Gamma(\mathbf{f})$ , we set

$$f_{i,\tau}(x) := \sum_{m \in \text{supp}(f_i) \cap \tau} c_{m,i} x^m.$$

**Definition 3.1.** (1) Let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be a nonconstant analytic mapping satisfying  $\mathbf{f}(0) = 0$ . The mapping  $\mathbf{f}$  is called *strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$* , if for any compact face  $\tau \subset \Gamma(\mathbf{f})$  and any  $z \in \{z \in (K^\times)^n \mid f_{1,\tau}(z) = \dots = f_{l,\tau}(z) = 0\}$  it verifies that  $\text{rank}_K \left[ \frac{\partial f_{i,\tau}}{\partial x_j}(z) \right] = \min\{l, n\}$ .

(2) Let  $\mathbf{f} = (f_1, \dots, f_l) : K^n \rightarrow K^l$  be a nonconstant polynomial mapping satisfying  $\mathbf{f}(0) = 0$ . The mapping  $\mathbf{f}$  is called *strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$* , if for any face  $\tau \subset \Gamma(\mathbf{f})$ , including  $\Gamma(\mathbf{f})$  itself, and any  $z \in \{z \in (K^\times)^n \mid f_{1,\tau}(z) = \dots = f_{l,\tau}(z) = 0\}$  it verifies that  $\text{rank}_K \left[ \frac{\partial f_{i,\tau}}{\partial x_j}(z) \right] = \min\{l, n\}$ .

**Remark 3.2.** Let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be a nonconstant analytic mapping satisfying  $\mathbf{f}(0) = 0$ .

(1) Let  $\gamma$  be a face of  $\Gamma(\mathbf{f})$  for which the rank condition in Definition 3.1 is satisfied. If  $\text{supp}(f_i) \cap \gamma \neq \emptyset \Leftrightarrow i \in I_\gamma$  for a non-empty subset  $I_\gamma \subseteq \{1, \dots, l\}$  satisfying  $\text{card}(I_\gamma) < \min\{l, n\}$ , then necessarily

$$\bigcap_{i \in I_\gamma} \{z \in (K^\times)^n \mid f_{i,\gamma}(z) = 0\} = \emptyset.$$

(2) If for a given face  $\gamma$  at least one  $f_{i,\gamma}$  is a monomial, then the rank condition on  $\gamma$  is satisfied. This is in particular true if  $\gamma$  is a point.

**Example 3.3.** Let  $\mathbf{f}(x, y) = (x^3 - xy, y)$ . The mapping  $\mathbf{f}$  is strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ , and also strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$ .

**Example 3.4.** Let  $\mathbf{f}(x, y, z) = (x^2, y^2, z^2, xy, xz, yz)$ . Then  $\mathbf{f}$  is strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ , and also strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$ ,

**3.2.1. Monomial mappings.** Any monomial mapping is strongly non-degenerate at the origin with respect to its Newton polyhedron. If  $\mathbf{f}_0$  is a fixed monomial mapping with Newton polyhedron  $\Gamma(\mathbf{f}_0)$ , and  $\mathbf{f} = \mathbf{f}_0 + \mathbf{g}$  is a deformation of  $\mathbf{f}_0$  such that all the monomials in  $\mathbf{g}$  have exponents in the interior of  $\Gamma(\mathbf{f}_0)$ , then  $\mathbf{f}$  is strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f}) = \Gamma(\mathbf{f}_0)$ . This type of mapping was introduced by the second author in [37, Definition 6.1]. Furthermore, the corresponding local zeta function can be computed by using a simple polyhedral subdivision subordinated to  $\Gamma(\mathbf{f}_0)$  [37, Theorem 6.1].

**3.2.2. Saia's non-degeneracy condition.** In [28] Saia introduced the following notion of non-degeneracy for ideals. Let  $I = (f_1, \dots, f_l)$  be a polynomial ideal.  $I$  is non-degenerate with respect to  $\Gamma(I)$  (where  $\Gamma(I) = \Gamma(\cup_{i=1}^l \text{supp}(f_i))$ ), if for every

compact face  $\tau$  of  $\Gamma(I)$ , the system of equations  $f_{1,\tau}(z) = 0, \dots, f_{l,\tau}(z) = 0$  does not have a solution in the torus  $(K^\times)^n$ . Thus Saia's notion of non-degeneracy is a particular case of our notion of non-degeneracy. Saia's notion of non-degeneracy plays an important role in the study of the integral closure of ideals.

**3.2.3. Khovanskii's non-degeneracy condition.** Now we discuss the relation between our notion of non-degeneracy and Khovanskii's notion of non-degeneracy of an analytic mapping with respect to several Newton polyhedra ([18], see also [26]). Given a positive vector  $a$  (i.e.  $a \in (\mathbb{N} \setminus \{0\})^n$ ), and an analytic mapping  $g$ , we set  $g_a(x) := g_{F(a)}(x)$ , where  $F(a)$  is the first meet locus of  $a$  with respect to  $\Gamma(g)$ . To make explicit the dependence between  $F(a)$  and  $\Gamma(g)$  we shall write  $F(a, \Gamma(g))$  instead of  $F(a)$ .

**Definition 3.5.** A nonconstant analytic mapping  $\mathbf{f} = (f_1, \dots, f_l) : U \longrightarrow K^l$ ,  $\mathbf{f}(0) = 0$ , is non-degenerate with respect to  $(\Gamma(f_1), \dots, \Gamma(f_l))$ , if for any positive vector  $a$  and any  $z \in \{z \in (K^\times)^n \mid f_{1,a}(z) = \dots = f_{l,a}(z) = 0\}$  it verifies that

$$\text{rank}_K \left[ \frac{\partial f_{i,a}}{\partial x_j}(z) \right] = \min\{l, n\}.$$

Here  $f_{j,a}(z) = f_{j, F(a, \Gamma(f_j))}(z)$  for every  $j$ .

The above definition is equivalent to the non-degeneracy notion given by Oka in [26], that is in turn a reformulation of the notion of non-degeneracy introduced by Khovanskii in [18].

**Remark 3.6.** Let  $\mathbf{f} = (f_1, \dots, f_l) : U \longrightarrow K^l$  be a nonconstant analytic mapping satisfying  $\mathbf{f}(0) = 0$ . Then  $\Gamma(\mathbf{f})$  is the convex hull in  $(\mathbb{R}_+)^n$  of  $\cup_{j=1}^l \Gamma(f_j)$ . This assertion follows from the fact that for any subsets  $A, B \subseteq (\mathbb{R}_+)^n$ ,  $\overline{A \cup B} = \overline{A} \cup \overline{B}$ , where the bar denotes the convex hull in  $(\mathbb{R}_+)^n$ .

The following is the relation between Khovanskii's non-degeneracy notion and the one introduced here.

**Proposition 3.7.** *Let  $\mathbf{f} = (f_1, \dots, f_l) : U \longrightarrow K^l$  be an analytic mapping strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ . Then  $\mathbf{f}$  is non-degenerate with respect to*

$$(\Gamma(f_1), \dots, \Gamma(f_l)).$$

*Proof.* Let  $a \in (\mathbb{N} \setminus \{0\})^n$  be a fixed positive vector. We set  $\Gamma = \Gamma(\mathbf{f})$ ,  $\Gamma_j = \Gamma(f_j)$ ,  $j = 1, \dots, l$ . Since  $\Gamma_j \subseteq \Gamma$  by the above remark,

$$d(a, \Gamma) = \min_{x \in \Gamma} \langle a, x \rangle \leq d(a, \Gamma_j) = \min_{x \in \Gamma_j} \langle a, x \rangle,$$

for  $j = 1, \dots, l$ . We define  $I \subseteq \{1, \dots, l\}$  by the condition

$$j \in I \Leftrightarrow d(a, \Gamma) = d(a, \Gamma_j).$$

Note that  $I \neq \emptyset$ . Then, if  $\tau := F(a, \Gamma)$ ,

$$F(a, \Gamma_j) \subseteq \tau, \text{ for } j \in I,$$

and

$$(3.2) \quad f_{j,\tau}(x) = \begin{cases} f_{j,a}(x), & j \in I, \\ 0, & j \in I^c. \end{cases}$$

If  $\text{card}(I) < \min\{l, n\}$ , then by Remark 3.2 the system of equations

$$f_{j,\tau}(x) = 0, j \in I, \text{ has no solutions in } (K^\times)^n.$$

Hence by using (3.2) the system of equations

$$f_{j,a}(x) = 0, j = 1, \dots, l, \text{ has no solutions in } (K^\times)^n,$$

and so the condition on  $a$  in Definition 3.5 is satisfied.

Now, we may assume that  $\text{card}(I) \geq \min\{l, n\}$ , and that

$$f_{j,\tau}(x) = 0, j \in I, \text{ has solutions in } (K^\times)^n.$$

Since  $\mathbf{f}$  is strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$ , it follows that

$$\text{rank}_K \left[ \frac{\partial f_{j,\tau}}{\partial x_i}(z) \right]_{\substack{j \in I \\ 1 \leq i \leq n}} = \min\{l, n\},$$

for any  $z \in \{z \in (K^\times)^n \mid f_{j,\tau}(z) = 0, j \in I\}$ . Then by (3.2),

$$\text{rank}_K \left[ \frac{\partial f_{j,a}}{\partial x_i}(z) \right]_{\substack{j \in I \\ 1 \leq i \leq n}} = \text{rank}_K \left[ \frac{\partial f_{j,\tau}}{\partial x_i}(z) \right]_{\substack{j \in I \\ 1 \leq i \leq n}} = \min\{l, n\},$$

for any  $z$  in

$$\{z \in (K^\times)^n \mid f_{j,a}(z) = 0, j \in I\} \supseteq \{z \in (K^\times)^n \mid f_{j,a}(z) = 0, j = 1, \dots, l\}.$$

Therefore,  $\mathbf{f}$  is non-degenerate in the sense of Khovanskii.  $\square$

**Example 3.8.** Let  $\mathbf{f}(x, y) = (x^2 - y^2, x^n, y^m)$ , with  $n, m \geq 3$ . Then  $\mathbf{f}$  is not strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ . Indeed,  $\Gamma(\mathbf{f})$  has only one compact facet,  $\tau$ , that is the straight segment from  $(0, 2)$  to  $(2, 0)$ . Then

$$\mathbf{f}_\tau(x, y) = (x^2 - y^2, 0, 0), \text{ and } \text{rank}_K \begin{bmatrix} 2z_1 & -2z_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = 1 \neq \min\{2, 3\},$$

for every  $(z_1, z_2) \in \{(z_1, z_2) \in (K^\times)^2 \mid z_1^2 - z_2^2 = 0\}$ , and therefore  $\mathbf{f}$  is not strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$ . On the other hand,  $\mathbf{f}$  is non-degenerate in the sense of Khovanskii.

### 3.3. Newton polyhedra and log-principalizations.

**Proposition 3.9.** *Let  $\mathbf{f} = (f_1, \dots, f_l) : U(\subseteq K^n) \longrightarrow K^l$  be a polynomial mapping (or more generally, an analytic mapping defined on  $U$ ) strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ . Let  $\mathcal{F}_\mathbf{f}$  be a simple fan subordinated to  $\Gamma(\mathbf{f})$ . Let  $Y_K$  be the toric manifold corresponding to  $\mathcal{F}_\mathbf{f}$ , and let*

$$\sigma_0 : Y_K \longrightarrow U$$

*be the restriction of the corresponding toric map to the inverse image of  $U$ . Denote by  $Z$  the set of common zeroes of  $\mathcal{I}_\mathbf{f} = (f_1, \dots, f_l)$  in  $U \cap (K^\times)^n$ . When  $U$  is taken small enough, either  $Z = \emptyset$  or it is a submanifold of codimension  $l$ . In this last case we have  $l < n$  and we denote the closure of  $Z$  in  $U$  and  $Y_K$  by  $Z_U$  and  $Z_Y$ , respectively.*

(1) *If  $Z = \emptyset$  (or if  $l = 1$ ), the ideal  $\sigma_0^*(\mathcal{I}_\mathbf{f})$  is principal (and monomial) in a sufficiently small neighborhood of  $\sigma_0^{-1}\{0\}$ .*

(2) If  $Z \neq \emptyset$ , we have that  $Z_Y$  is a closed submanifold of  $Y_K$ , having normal crossings with the exceptional divisor of  $\sigma_0$ . Let  $\sigma_1 : X_K \rightarrow Y_K$  be the blowing-up of  $Y_K$  with center  $Z_Y$ , and let  $\sigma = \sigma_0 \circ \sigma_1 : X_K \rightarrow U$ . Then the ideal  $\sigma^*(\mathcal{I}_{\mathbf{f}})$  is principal (and monomial) in a sufficiently small neighborhood of  $\sigma^{-1}\{0\}$ .

*Proof.* We first recall the construction of  $(Y_K, \sigma_0)$  from a simple fan  $\mathcal{F}_{\mathbf{f}}$  subordinated to  $\Gamma(\mathbf{f})$  (see e.g. [2]). Let  $\Delta_{\tau}$  be an  $n$ -dimensional simple cone in  $\mathcal{F}_{\mathbf{f}}$  such that  $F(a) = \tau$  for any  $a \in \Delta_{\tau}$ . Then the face  $\tau$  of  $\Gamma(\mathbf{f})$  is necessarily a point. Let  $a_1, \dots, a_n$  be the generators of  $\Delta_{\tau}$ . Then in the chart of  $Y_K$  corresponding to  $\Delta_{\tau}$ , the map  $\sigma_0$  has the form

$$(3.3) \quad \begin{array}{ccc} \sigma_0 : & K^n & \longrightarrow U \\ & y & \longrightarrow x, \end{array}$$

where  $x_i = \prod_j y_j^{a_{i,j}}$ , with  $[a_{i,j}] = [a_1, \dots, a_n]$ . Denote this chart by  $V_{\tau}$ . We slightly abuse notation here : since  $\sigma_0$  only maps to  $U$  instead of to the whole of  $K^n$ , at some charts it will not be defined everywhere on  $K^n$ . If  $f_i(x) = \sum_m c_{m,i} x^m$  for  $i = 1, \dots, l$ , then

$$(f_i \circ \sigma_0)(y) = \sum_m c_{m,i} \prod_{j=1}^n y_j^{\langle m, a_j \rangle} \text{ for } i = 1, \dots, l.$$

If  $\text{supp}(f_i) \cap \tau \neq \emptyset$ , then the minimum of all  $\langle m, a_j \rangle$  is attained at  $\tau$ , and then

$$(3.4) \quad (f_i \circ \sigma_0)(y) = \left( \prod_{j=1}^n y_j^{d(a_j)} \right) \tilde{f}_i(y), \text{ with } \tilde{f}_i(0) \neq 0$$

(cf. [2, page 201, Lemma 8]). If  $\text{supp}(f_i) \cap \tau = \emptyset$ ,

$$(3.5) \quad (f_i \circ \sigma_0)(y) = \left( \prod_{j=1}^n y_j^{d(a_j)} \right) \tilde{f}_i(y), \text{ with } \tilde{f}_i(0) = 0.$$

Then, from (3.4) and (3.5), we have in a neighborhood of the origin of  $V_{\tau}$  that  $\sigma_0^*(\mathcal{I}_{\mathbf{f}})$  is generated by  $\prod_{j=1}^n y_j^{d(a_j)}$ .

Now let us consider on  $V_{\tau}$  the points on  $\sigma_0^{-1}(0)$ , different from the origin of  $V_{\tau}$ . We will study simultaneously points with exactly  $r$  zero coordinates (where  $1 \leq r \leq n-1$ ); after permuting indices, we may assume that the first  $r$  coordinates are zero.

Let  $\tau'$  be the first meet locus of the cone  $\Delta_{\tau'}$  spanned by  $a_1, \dots, a_r$ ; it is a compact face of  $\Gamma(\mathbf{f})$  (cf. [2, page 201, Lemma 8]). We can write  $(f_i \circ \sigma_0)(y)$  as

$$(3.6) \quad (f_i \circ \sigma_0)(y) = \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \left( \tilde{f}_i(y_{r+1}, \dots, y_n) + O_i(y_1, \dots, y_n) \right),$$

where the  $\tilde{f}_i$  are polynomials in  $y_{r+1}, \dots, y_n$ , and the  $O_i(y_1, \dots, y_n)$  are analytic functions in  $y_1, \dots, y_n$  but belonging to the ideal generated by  $y_1, \dots, y_r$ . Here the  $\tilde{f}_i$  are identically zero if and only if  $\text{supp}(f_i) \cap \tau' = \emptyset$ . Furthermore,

$$(3.7) \quad (f_{i,\tau'} \circ \sigma_0)(y) = \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \tilde{f}_i(y_{r+1}, \dots, y_n).$$

We investigate the  $(f_i \circ \sigma_0)(y)$  for  $p = (0, \dots, 0, p_{r+1}, \dots, p_n)$  with

$$\tilde{p} = (p_{r+1}, \dots, p_n) \in (K^\times)^{n-r}.$$

We have to study two cases. The first case occurs when there exists an index  $i$  such that  $\tilde{f}_i(\tilde{p}) \neq 0$ . In this case as before  $\sigma_0^*(\mathcal{I}_{\mathbf{f}})$  is generated by  $\prod_{j=1}^r y_j^{d(a_j)}$  in a neighborhood of  $p$ .

The second case occurs when  $\tilde{f}_i(\tilde{p}) = 0$ , for all  $i = 1, \dots, l$ . We recall that, by the non-degeneracy condition,  $\text{rank}_K \left[ \frac{\partial f_{i,\tau'}}{\partial x_j}(x) \right] = \min\{l, n\}$  for  $x \in (K^\times)^n \cap \{f_{1,\tau'}(x) = \dots = f_{l,\tau'}(x) = 0\}$ . Since  $\sigma_0$  is an isomorphism over  $(K^\times)^n$ , then also  $\text{rank}_K \left[ \frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y) \right] = \min\{l, n\}$  for  $y \in (K^\times)^n \cap \{f_{1,\tau'}(\sigma_0(y)) = \dots = f_{l,\tau'}(\sigma_0(y)) = 0\}$ . Note that by (3.7) this condition on  $y$  is equivalent to  $y \in (K^\times)^n \cap \{\tilde{f}_1(y) = \dots = \tilde{f}_l(y) = 0\}$  and that  $\left[ \frac{\partial f_{i,\tau'} \circ \sigma_0}{\partial y_j}(y) \right]$  for such  $y$  is equal to

$$\begin{pmatrix} 0 & \dots & 0 & \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \frac{\partial \tilde{f}_1}{\partial y_{r+1}}(y) & \dots & \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \frac{\partial \tilde{f}_1}{\partial y_n}(y) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \frac{\partial \tilde{f}_l}{\partial y_{r+1}}(y) & \dots & \left( \prod_{j=1}^r y_j^{d(a_j)} \right) \frac{\partial \tilde{f}_l}{\partial y_n}(y) \end{pmatrix}.$$

Now this implies that for  $\tilde{y} = (y_{r+1}, \dots, y_n) \in (K^\times)^{n-r} \cap \{\tilde{f}_1(\tilde{y}) = \dots = \tilde{f}_l(\tilde{y}) = 0\}$  the rank of the matrix

$$\begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial y_{r+1}}(\tilde{y}) & \dots & \frac{\partial \tilde{f}_1}{\partial y_n}(\tilde{y}) \\ \dots & \dots & \dots \\ \frac{\partial \tilde{f}_l}{\partial y_{r+1}}(\tilde{y}) & \dots & \frac{\partial \tilde{f}_l}{\partial y_n}(\tilde{y}) \end{pmatrix}$$

is equal to  $\min\{l, n\}$ . Then necessarily the rank is  $l$ , and we must have that  $l \leq n-r$ .

So when  $p$  above satisfies  $\tilde{f}_i(\tilde{p}) = 0$  for  $i = 1, \dots, l$ , then necessarily all  $\tilde{f}_i$  are nonzero polynomials,  $r \leq n-l$ , and  $\text{rank}_K \left[ \frac{\partial \tilde{f}_i}{\partial y_j}(\tilde{p}) \right] = l$ . Now  $\left[ \frac{\partial \tilde{f}_i}{\partial y_j}(\tilde{p}) \right] = \left[ \frac{\partial (\tilde{f}_i + O_i)}{\partial y_j}(p) \right]$  (cf. (3.6)). This last matrix having rank  $l$  implies that we can choose new coordinates  $y' = (y_1, \dots, y_r, y'_{r+1}, \dots, y'_n)$  in a neighborhood  $V_p$  of  $p$  such that

$$(3.8) \quad (f_i \circ \sigma_0)(y') = \left( \prod_{j=1}^r y_j^{d(a_j)} \right) y'_{r+i} \text{ for } i = 1, \dots, l.$$

Since  $\sigma_0$  is an isomorphism on  $(K^\times)^n$ , we have that  $\{y'_{r+1} = \dots = y'_{r+l} = 0\}$  is the description in  $V_p$  of  $Z_Y \subset Y$ . (The local description (3.8) yields that  $Z$  is a submanifold of  $(K^\times)^n$  of codimension  $l$ .) Clearly  $Z_Y$  is a submanifold of  $Y$  of codimension  $l$ , having normal crossings with the exceptional divisor of  $\sigma_0$ .

So,  $\sigma_1$  being the blowing-up of  $Y$  in  $Z_Y$ , we obtain by (3.8) that  $(\sigma_0 \circ \sigma_1)^*(\mathcal{I}_{\mathbf{f}})$  becomes principal.  $\square$

**Remark 3.10.** If we replace in Proposition 3.9 the condition *strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$*  by the condition *strongly non-degenerate with respect to  $\Gamma(\mathbf{f})$* , and  $U$  by  $K^n$ , with a similar proof we obtain a *global* version of the proposition, that is, the conclusions (1) and (2) are valid without the condition *in a sufficiently small neighborhood*. In this case  $Z_Y$  may have components that are disjoint with the exceptional divisor of  $\sigma_0$ .

Given  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{N}^n \setminus \{0\}$ , we put  $\sigma(\xi) := \xi_1 + \dots + \xi_n$  and  $d(\xi) = \min_{x \in \Gamma(\mathbf{f})} \langle \xi, x \rangle$  as before. We say that  $\xi$  is a primitive vector, if  $\gcd(\xi_1, \dots, \xi_n) = 1$ . If  $d(\xi) \neq 0$ , we define

$$\mathcal{P}(\xi) = \left\{ -\frac{\sigma(\xi)}{d(\xi)} + \frac{2\pi\sqrt{-1}k}{d(\xi)\log q}, k \in \mathbb{Z} \right\}.$$

Let  $\mathcal{F}_{\mathbf{f}}$  be a simple fan subordinated to  $\Gamma(\mathbf{f})$ . Then the set of generators of the cones in  $\mathcal{F}_{\mathbf{f}}$ , i.e. the skeleton of  $\mathcal{F}_{\mathbf{f}}$ , can be partitioned as  $\Lambda_{\mathbf{f}} \cup \mathfrak{D}(\Gamma(\mathbf{f}))$ , where  $\Lambda_{\mathbf{f}}$  is a finite set of primitive vectors, corresponding to the extra rays, induced by the subdivision into simple cones.

The numerical data of the log-principalizations constructed in Proposition 3.9 and Remark 3.10 can be computed directly from the explicit expressions for the generators of  $\sigma_0^*(I_{\mathbf{f}})$ ,  $\sigma^*(I_{\mathbf{f}})$ , and Lemma 8 in [2, page 201]. Then Theorem 2.4 yields that the poles of  $Z_{\Phi}(s, \mathbf{f})$  belong to the set

$$(3.9) \quad \bigcup_{\xi \in \Lambda_{\mathbf{f}}} \mathcal{P}(\xi) \cup \bigcup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, k \in \mathbb{Z} \right\},$$

where the last set may be discarded if  $l \geq n$ .

This provides a generalization to the case  $l \geq 1$  of a well-known result that describes the poles of the local zeta function associated to a non-degenerate polynomial in terms of the corresponding Newton polyhedron [21], [5], [7], [36]. This result was originally established by Varchenko [31] for local zeta functions over  $\mathbb{R}$ . As in the case  $l = 1$ , the list (3.9) is too big. More precisely, the set  $\bigcup_{\xi \in \Lambda_{\mathbf{f}}} \mathcal{P}(\xi)$  is not necessary. This fact is established by analogous arguments as in [5] where the case  $l = 1$  is studied.

**Theorem 3.11.** (1) Let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be an analytic mapping strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ . If  $U$  is a sufficiently small neighborhood of the origin, and  $\Phi$  is a Schwartz-Bruhat function whose support is contained in  $U$ , then the poles of  $Z_{\Phi}(s, \mathbf{f})$  belong to the set  $\bigcup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, k \in \mathbb{Z} \right\}$ , where the last set may be discarded if  $l \geq n$ .  
(2) If  $\mathbf{f} : K^n \rightarrow K^l$  is a strongly non-degenerate polynomial mapping with respect to  $\Gamma(\mathbf{f})$ , then the poles of  $Z(s, \mathbf{f})$  belong to the set

$$\bigcup_{\xi \in \mathfrak{D}(\Gamma(\mathbf{f}))} \mathcal{P}(\xi) \cup \left\{ -l + \frac{2\pi\sqrt{-1}k}{\log q}, k \in \mathbb{Z} \right\}.$$

The above result can be restated in a geometric form as follows. If  $s$  is a pole of  $Z_{\Phi}(s, \mathbf{f})$ , then  $\operatorname{Re}(s)$  is  $-l$ , or  $\operatorname{Re}(s)$  is of the form  $-1/t_0$ , where  $(t_0, \dots, t_0)$  is the intersection point of the diagonal  $\{(t, \dots, t) \in \mathbb{R}^n\}$  with the supporting hyperplane of a facet of  $\Gamma(\mathbf{f})$ .

By using Theorems 2.7 and 3.11 we obtain the following corollary.

**Corollary 3.12.** (1) Let  $U$  be a sufficiently small neighborhood of the origin, and let  $\mathbf{f} = (f_1, \dots, f_l) : U \rightarrow K^l$  be an analytic mapping strongly non-degenerate at the origin with respect to  $\Gamma(\mathbf{f})$ . Let  $(t_{\mathbf{f}}, \dots, t_{\mathbf{f}}) \in \mathbb{Q}^n$  be the intersection point of the diagonal  $\{(t, \dots, t) \in \mathbb{R}^n\}$  with the boundary of  $\Gamma(\mathbf{f})$ . If  $t_{\mathbf{f}} \geq 1/l$ , then  $-1/t_{\mathbf{f}}$  is the largest real part of a pole of  $Z_U(s, \mathbf{f})$ .

(2) Let  $\mathbf{f} : K^n \rightarrow K^l$  be a strongly non-degenerate polynomial mapping with respect to  $\Gamma(\mathbf{f})$ . If  $t_{\mathbf{f}} \geq 1/l$ , then  $-1/t_{\mathbf{f}}$  is the largest real part of a pole of  $Z(s, \mathbf{f})$ .

The largest real part of the poles of  $Z(s, \mathbf{f})$ ,  $l = 1$ , when  $\mathbf{f}$  is non-degenerate with respect to its Newton polyhedron  $\Gamma(\mathbf{f})$  and  $t_{\mathbf{f}} > 1$  follows from observations made by Varchenko in [31] and was originally noted in the  $p$ -adic case in [21]. The case  $t_{\mathbf{f}} = 1$  is treated in [7]. The case of  $t_{\mathbf{f}} < 1$  is more difficult and is established in [7] with some additional conditions on  $\Gamma(\mathbf{f})$  by using a difficult result on exponential sums. In [36] the second author established the case  $t_{\mathbf{f}} \geq 1$  when  $\mathbf{f}$  is a non-degenerate polynomial with coefficients in a non-archimedean local field of arbitrary characteristic.

#### 4. EXPLICIT FORMULAS AND NEWTON POLYHEDRA

In [7, Theorem 4.2] Denef and Hoornaert gave an explicit formula for  $Z(s, \mathbf{f})$ ,  $l = 1$ , associated to a polynomial  $\mathbf{f}$  in several variables over the  $p$ -adic numbers, when  $\mathbf{f}$  is sufficiently non-degenerate with respect to its Newton polyhedron  $\Gamma(\mathbf{f})$ . This explicit formula can be generalized to the case  $l \geq 1$  by using the condition of non-degeneracy for polynomial mappings introduced in this paper.

Let as before  $K$  be a  $p$ -adic field with valuation ring  $R_K$ , maximal ideal  $P_K$  and residue field  $\overline{K} = \mathbb{F}_q$ . For any polynomial  $g$  over  $R_K$  we denote by  $\overline{g}$  the polynomial over  $\overline{K}$  obtained by reducing each coefficient of  $g$  modulo  $P_K$ .

**Definition 4.1.** Let  $f_i \in R_K[x]$ ,  $x = (x_1, \dots, x_n)$ , satisfying  $f_i(0) = 0$  for  $i = 1, \dots, l$ . The mapping  $\mathbf{f} = (f_1, \dots, f_l) : K^n \rightarrow K^l$  is called *strongly non-degenerate over  $\overline{K}$  with respect to  $\Gamma(\mathbf{f})$* , if for any face  $\tau$  of  $\Gamma(\mathbf{f})$ , including  $\Gamma(\mathbf{f})$  itself, we have that  $\text{rank}_K \left[ \frac{\partial \overline{f_{i,\tau}}}{\partial x_j}(\overline{z}) \right] = \min\{l, n\}$ , for any  $\overline{z} \in (\overline{K}^\times)^n$  satisfying  $\overline{f_{1,\tau}}(\overline{z}) = \dots = \overline{f_{l,\tau}}(\overline{z}) = 0$ . Analogously we call  $\mathbf{f}$  *strongly non-degenerate at the origin over  $\overline{K}$  with respect to  $\Gamma(\mathbf{f})$* , if the same condition is satisfied but only for the compact faces  $\tau$  of  $\Gamma(\mathbf{f})$ .

**Theorem 4.2.** (1) Let  $\mathbf{f} = (f_1, \dots, f_l) : K^n \rightarrow K^l$  be a strongly non-degenerate polynomial mapping over  $\overline{K}$ . Denote for each face  $\tau$  of  $\Gamma(\mathbf{f})$ , including  $\Gamma(\mathbf{f})$  itself,

$$\overline{D}_\tau := \left\{ \overline{x} \in (\overline{K}^\times)^n \mid \overline{f_{1,\tau}}(\overline{x}) = \dots = \overline{f_{l,\tau}}(\overline{x}) = 0 \right\}.$$

Fix a rational simplicial polyhedral subdivision  $\{\Delta_{\tau,i}\}$ , with  $\tau$  a proper face, subordinated to  $\Gamma(\mathbf{f})$  as in (3.1). Denote by  $a_j$ ,  $j = 1, \dots, r_{\Delta_{\tau,i}}$ , the generators of the cone  $\Delta_{\tau,i}$ . Then

$$Z(s, \mathbf{f}) = L_{\Gamma(\mathbf{f})}(q^{-s}) + \sum_{\tau \neq \Gamma(\mathbf{f})} L_\tau(q^{-s}) \left( \sum_i S_{\tau,i}(q^{-s}) \right).$$

Here

$$L_\tau(q^{-s}) = q^{-n} \left( (q-1)^n - \frac{\text{card}(\overline{D}_\tau)(1-q^{-s})}{1-q^{-\min\{l,n\}-s}} \right),$$

for each face  $\tau$  of  $\Gamma(\mathbf{f})$ , including  $\Gamma(\mathbf{f})$ , and

$$S_{\tau,i}(q^{-s}) = \frac{\left(\sum_h q^{\sigma(h)+d(h)s}\right) q^{-\sum_{j=1}^{r_{\Delta_{\tau,i}}} (\sigma(a_j)+d(a_j)s)}}{\prod_{j=1}^{r_{\Delta_{\tau,i}}} (1 - q^{-\sigma(a_j)-d(a_j)s})},$$

where  $h$  runs through the elements of the set

$$\mathbb{Z}^n \cap \left\{ \sum_{j=1}^{r_{\Delta_{\tau,i}}} \lambda_j a_j \mid 0 \leq \lambda_j < 1 \text{ for } j = 1, \dots, r_{\Delta_{\tau,i}} \right\}.$$

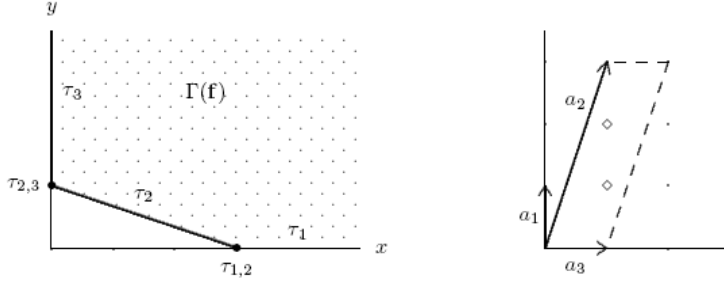
(2) With the same notations and only assuming that  $\mathbf{f}$  is strongly non-degenerate at the origin over  $\overline{K}$  we have

$$Z_0(s, \mathbf{f}) = \sum_{\tau \text{ compact}} L_{\tau}(q^{-s}) \left( \sum_i S_{\tau,i}(q^{-s}) \right).$$

The proof of the above result is analogous to the case  $l = 1$  treated in [7, Theorem 4.2].

By using a simple polyhedral subdivision one obtains a slightly less complicated explicit formula in which all the terms  $\sum_h q^{\sigma(h)+d(h)s}$  are identically 1. But then in general we have to introduce new rays which give rise to superfluous candidate poles.

**Example 4.3.** Let  $\mathbf{f} = (x^3 - xy, y)$  as in Example 2.14. It is strongly non-degenerate over  $\overline{K}$  with respect to  $\Gamma(\mathbf{f})$ .



We shall compute  $Z(s, \mathbf{f})$  using Theorem 4.2 and the obvious rational simplicial polyhedral subdivision of  $\mathbb{R}_+^2$ . More precisely, set  $a_1 = (0, 1)$ ,  $a_2 = (1, 3)$ , and  $a_3 = (1, 0)$ ;  $\Delta_i = \{\lambda a_i \mid \lambda > 0\}$  for  $i = 1, 2, 3$ , and  $\Delta_{i,i+1} = \{\lambda a_i + \lambda' a_{i+1} \mid \lambda, \lambda' > 0\}$ ,  $i = 1, 2$ . Then

$$\mathbb{R}_+^2 = \{0\} \cup \Delta_1 \cup \Delta_{1,2} \cup \Delta_2 \cup \Delta_{2,3} \cup \Delta_3.$$

With the notation of Theorem 4.2 one easily verifies that all  $\overline{D}_{\tau} = \emptyset$  and hence all  $L_{\tau} = q^{-2}(q-1)^2$ . Further

$$S_{\tau_1} = S_{\tau_3} = \frac{q^{-1}}{1 - q^{-1}}, \quad S_{\tau_2} = \frac{q^{-4-3s}}{1 - q^{-4-3s}},$$

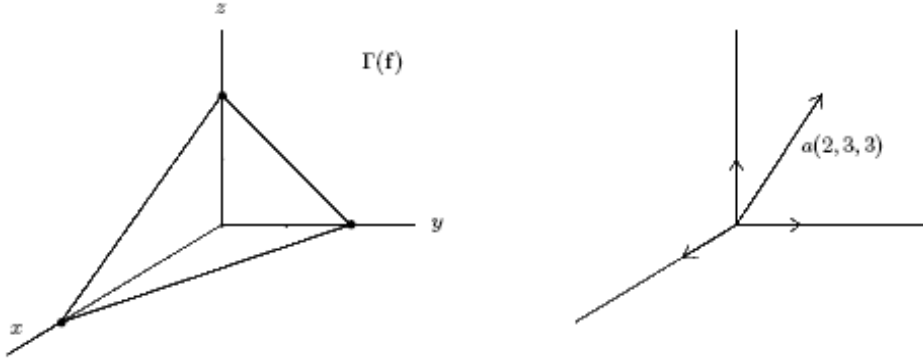
$$S_{\tau_{1,2}} = \frac{q^{-5-3s}}{(1-q^{-1})(1-q^{-4-3s})}, \quad S_{\tau_{2,3}} = \frac{(1+q^{2+s}+q^{3+2s})q^{-5-3s}}{(1-q^{-1})(1-q^{-4-3s})}.$$

Therefore

$$Z(s, \mathbf{f}) = q^{-2}(q-1) \frac{(q+1+q^{-1-s}+q^{-2-2s})}{1-q^{-4-3s}}.$$

If we would use the natural *simple* polyhedral subdivision of the one above, introducing two new rays generated by  $(1, 1)$  and  $(1, 2)$ , we would introduce the same superfluous (real) candidate poles  $-2$  and  $-\frac{3}{2}$  as in Example 2.14. This is reasonable because the log-principalization of Proposition 3.9 associated to this simple fan is in fact the same as the one constructed in Example 2.14.

**Example 4.4.** Let  $\mathbf{f} = (y^2 - x^3, y^2 - z^2)$  as in Example 2.15. When  $\text{char}(\overline{K}) \neq 2$ , it is strongly non-degenerate at the origin over  $\overline{K}$  with respect to  $\Gamma(\mathbf{f})$ .



The Newton polyhedron  $\Gamma(\mathbf{f})$  has seven compact faces. The polyhedral subdivision associated to it is already simplicial, so in the formulation of Theorem 4.2 (2) we need to sum over seven cones: the ray through  $a = (2, 3, 3)$ , the three 2-dimensional cones with  $a$  in their boundaries, and the three 3-dimensional cones. We note that all the  $\overline{D}_\tau = \emptyset$ , except when  $\tau$  is the unique compact facet, in this case  $\text{card}(\overline{D}_\tau) = 2(q-1)$ . Concerning the  $S_\tau(q^{-s})$  we just mention that the expression  $\sum_h q^{\sigma(h)+d(h)s}$  is three times equal to 1, three times equal to  $1+q^{3+2s}+q^{6+4s}$ , and once to  $1+q^{5+3s}$ . One can verify that the formula in Theorem 4.2 yields the same expression for  $Z_0(s, \mathbf{f})$  as in Example 2.15. Note that  $-8/6$  and  $-2$  are the only (real) candidate poles given by Theorems 3.11 or 4.2.

**Remark 4.5.** With the obvious analogous definitions for strongly non-degeneracy over  $\mathbb{C}$ , we have the following. Suppose that  $f_1, \dots, f_l$  are polynomials in  $n$  variables with coefficients in a number field  $F (\subseteq \mathbb{C})$ . Then we can consider  $\mathbf{f} = (f_1, \dots, f_l)$  as a map  $K^n \rightarrow K^l$  for any non-archimedean completion  $K$  of  $F$ . If  $\mathbf{f}$  is strongly non-degenerate at the origin over  $\mathbb{C}$  with respect to  $\Gamma(\mathbf{f})$ , then  $\mathbf{f}$  is strongly non-degenerate over  $\overline{K}$  with respect to  $\Gamma(\mathbf{f})$  for almost all the completions  $K$  of  $F$ . (And analogously for non-degeneracy at the origin.) This fact follows by applying the Weak Nullstellensatz.

**Remark 4.6.** By using our notion of non-degeneracy with respect to a Newton polyhedron it is also possible to give lists of candidate poles and explicit formulae for the motivic and topological zeta functions introduced in 2.4, associated to

a polynomial ideal. These explicit formulas are reasonably straightforward generalizations of those in [1] and [10, Théorème 5.3 (i)]. For the topological zeta function one requires here strongly non-degeneracy with respect to all the faces of the “global” Newton polyhedron as in [10, (5.1)].

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