On the smallest poles of Igusa's p-adic zeta functions

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Abstract

Let K be a p-adic field. We explore Igusa's p-adic zeta function, which is associated to a K-analytic function on an open and compact subset of K^n. First we deduce a formula for an important coefficient in the MacLaurin series of this meromorphic function at a candidate pole. Afterwards we use this formula to determine all values less than $-1/2$ for $n = 2$ and less than -1 for $n = 3$ which occur as the real part of a pole.

1 Introduction

(1.1) Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z and $|z| = q^{-\operatorname{ord} z}$ the absolute value of z.

(1.2) Let f be a K-analytic function on an open and compact subset X of K^n and put $x = (x_1, \ldots, x_n)$. Igusa's p-adic zeta function of f is defined by

$$Z_f(s) = \int_X |f(x)|^s \, |dx|$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq 0$, where $|dx|$ denotes the Haar measure on K^n, so normalised that R^n has measure 1. Igusa proved that it is a rational function of q^{-s}, so that it extends to a meromorphic function $Z_f(s)$ on $\mathbb{C}$ which is also called Igusa's p-adic zeta function of f.

(1.3) This zeta function has an interesting connection with number theory. Let f be a K-analytic function on R^n defined by a power series over R which is convergent on the whole of R^n. Let M_i be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$. All the M_i's are described by $Z_f(s)$ through the relation

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$$Z_f(s) = (1 - q^s)P(q^{-s}) + q^s,$$

where the Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} M_i (q^{-n}t)^i.$$

Remark that $P(t)$ is a rational function of t because $Z_f(s)$ is a rational function of q^{-s}.

(1.4) The poles of $Z_f(s)$ are an interesting object of study because they are related to the monodromy conjecture [De2, (2.3.2)] and because they determine the asymptotic behaviour of the M_i. The poles with largest real part give the largest contribution to the M_i. In this paper we are concerned with the smallest real part l of a pole of $Z_f(s)$. A nontrivial consequence of the fact that the M_i are integers is that l is larger than or equal to $-n$. Our main results are stated in the next paragraph and sharpen this bound by using a completely different method. This better bound has number theoretic consequences because the knowledge of l gives us interesting information about the M_i: there exists an $a \in \mathbb{Z}$ such that M_i is divisible by $q^{\ulcorner (n+l)i-a \urcorner}$ for all i (for which $(n+l)i - a \geq 0$). This is proved in the appendix. Remark that a is independent of i and that the number in the exponent is the smallest integer larger than or equal to $(n+l)i - a$.

Let F_n^K denote the set of all K-analytic functions defined on an arbitrary open and compact subset of K^n. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}_n^K$ by

$$\mathcal{P}_n^K := \{s_0 \mid \exists f \in F_n^K \ : \ Z_f(s) \textit{ has a pole with real part } s_0\}.$$

In this article, we will prove that

$$\begin{aligned}\mathcal{P}_2^K \cap]-\infty, -1/2[\ &= \ \{-1/2 - 1/i \mid i \in \mathbb{Z}_{>1}\} \\ &= \ \{-1, -5/6, -3/4, -7/10, \ldots\}\end{aligned}$$

and that

$$\mathcal{P}_3^K \cap]-\infty, -1[\ = \ \{-1 - 1/i \mid i \in \mathbb{Z}_{>1}\}.$$

In general, we expect that $\mathcal{P}_n^K \cap]-\infty, -(n-1)/2[= \{-(n-1)/2 - 1/i \mid i \in \mathbb{Z}_{>1}\}$.

Remark. One can easily show that $\mathcal{P}_n^K \cap]-\infty, -n+1[= \emptyset$ if $n \geq 2$.

(1.5) Let $f \in K[x_1, x_2]$. Consider f as a polynomial over $K^{\mathrm{alg\,cl}}$. Suppose that the minimal embedded resolution g of $f^{-1}\{0\} \subset (K^{\mathrm{alg\,cl}})^2$ is defined over K, i.e., all irreducible components of $g^{-1}(f^{-1}\{0\})$ over $K^{\mathrm{alg\,cl}}$ and all points in the intersection of two such components are defined over K. Then it is generally known that an exceptional curve which is intersected once or twice does not contribute to the residues of its candidate poles with candidate order 1. Because $K^{\text{alg cl}} \cong \mathbb{C}$, we can use the calculations in [SV] to conclude that the real part of a pole of $Z_f(s)$ is of the form $-1/2 - 1/i$, $i \in \mathbb{Z}_{>1}$, if it is smaller than $-1/2$.

Let $f \in K[x_1, x_2, x_3]$. Consider f again as a polynomial over $K^{\text{alg cl}} \cong \mathbb{C}$. Suppose that there exists an embedded resolution g of $f^{-1}\{0\} \subset (K^{\text{alg cl}})^3 \simeq \mathbb{C}^3$ for which the induced embedded resolution of the germ at each point P of $\mathbb{C}^3$ satisfies the conditions in [SV, (3.1.1)], which is defined over K and which has good reduction modulo P (see [De1, section 2]). Then the vanishing results in [Ve3] and the calculations in [SV] imply that the real part of a pole of $Z_f(s)$ is of the form $-1 - 1/i$, $i \in \mathbb{Z}_{>1}$, if it is smaller than -1.

Consequently, starting from [SV], it is rather easy to deal with polynomials which allow an appropriate embedded resolution. However it is very difficult to verify the existence of such an embedded resolution for a concrete function f. In a lot of cases there does not exist an embedded resolution which is defined over K, and if it exists, the condition of good reduction modulo P is very hard to check. This gives us a strong motivation to study the general case. In this article there are no constraints on f: we will not require that g is defined over K and that g has good reduction modulo P.

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2 The tool for our vanishing results

(2.1) Let K be a p-adic field. Let $x = (x_1, \dots, x_n)$ be the coordinates of K^n. Let f be a K-analytic function on an open and compact subset X of K^n. An embedded resolution $g : Y \to X$ of (f, dx) consists of a K-analytic manifold Y, a proper K-analytic map g and a finite set $\{E_i \mid i \in T\}$ of closed submanifolds of Y of codimension one with a pair of positive integers (N_i, ν_i), called numerical data, assigned to each E_i such that

1. the union of the E_i is equal to $g^{-1}(f^{-1}\{0\})$,
2. the restriction $Y \setminus g^{-1}(f^{-1}\{0\}) \to X \setminus f^{-1}\{0\}$ is a K-bianalytic map,
3. for every point b of Y, if $E_1, \dots, E_k$ are all the E_i that contain b, there exists a chart $(V, y = (y_1, \dots, y_n))$ around b with y_i, $1 \leq i \leq k$, an equation of E_i on V such that

$$f \circ g = \varepsilon \prod_{i=1}^{k} y_i^{N_i} \text{ and } g^* dx = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$$

on V for nonvanishing K-analytic functions ε and η on V.

Let Y be a n-dimensional K-analytic manifold, ω a K-analytic differential n-form on Y and h a K-analytic function on an open subset U of Y. We say that a chart (V, y) on Y is a good chart for (h, ω) if $V \subset U$, $h = \varepsilon \prod_{i=1}^{k} y_i^{N_i}$ on V and $\omega = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$ on V for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$, $\nu_i \in \mathbb{Z}_{>0}$ and nonvanishing K-analytic functions ε, η on V. We say that (h, ω) has normal crossings at a point $P \in U$ if there exists a good chart for (h, ω) around P. So when we say normal crossings, we mean normal crossings over K.

Let f be a K-analytic function on an open and compact subset X of K^n. Let Y be an n-dimensional K-analytic manifold and $g : Y \to X$ a K-analytic map which is a composition of blowing-ups along K-analytic closed submanifolds which are contained in the zero locus of the pullback of f. In this situation we have the following. If $y = (y_1, \ldots, y_n)$ is a system of local parameters at $P \in Y$ such that $f \circ g = \varepsilon \prod_{i=1}^{k} y_i^{N_i}$ for $k \in \{0, \ldots, n\}$, $N_i \in \mathbb{Z}_{>0}$ and ε a unit in the local ring at P, then $g^* dx = \eta \prod_{i=1}^{k} y_i^{\nu_i - 1} dy$ for $\nu_i \in \mathbb{Z}_{>0}$ and η a unit in the local ring at P. Consequently we will talk in this context about an embedded resolution of f instead of (f, dx) and about normal crossings of $f \circ g$ at P instead of $(f \circ g, g^* dx)$. Although the condition on $f \circ g$ does not imply the condition on $g^* dx$ globally, we will talk about a good chart for $f \circ g$ instead of $(f \circ g, g^* dx)$ in this context.

(2.2) Let $g : Y \to X$ be an embedded resolution of (f, dx). We study Igusa's p-adic zeta function $Z_f(s)$ by calculating the integral on the resolution Y:

$$\begin{aligned} Z_f(s) &= \int_X |f(x)|^s |dx| \\ &= \int_Y |f \circ g|^s |g^* dx|. \end{aligned}$$

Because $|\varepsilon|$ and $|\eta|$ are locally constant functions on each chart and because Y is a compact K-analytic manifold, we can choose a finite set J of good charts (V, y) for $(f \circ g, g^* dx)$ such that $|\varepsilon|$ and $|\eta|$ are constant on each chart, the V's form a partition of Y and for each chart (V, y) we have $y(V) = P^j := P^{j_1} \times \cdots \times P^{j_n}$ for some $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$ depending on (V, y). Remark that we may even require that $j_1 = \cdots = j_n$ and that this value does not depend on the chart, but we will not do this. Because Y is a finite disjoint union of the V's, we obtain

$$Z_f(s) = \sum_{(V,y) \in J} \int_V |(f \circ g)(y)|^s |g^* dx|.$$

We can calculate these integrals very explicitly because we know in the one variable case that

$$\int_{P^j} |x|^{\alpha - 1} |dx| = \frac{q-1}{q} \frac{q^{-j\alpha}}{1 - q^{-\alpha}}$$

for a complex number α with $\mathrm{Re}(\alpha) > 0$ (the integral is not defined if $\mathrm{Re}(\alpha) \leq 0$) and because $|\varepsilon|$ and $|\eta|$ are constant on each chart:

$$\begin{aligned}
\int_V |(f \circ g)(y)|^s |g^* dx| &= \int_V |\varepsilon|^s |\eta| \prod_{i=1}^{k} |y_i|^{N_i s + \nu_i - 1} |dy| \\
&= \int_{P^j} |\varepsilon|^s |\eta| \prod_{i=1}^{k} |y_i|^{N_i s + \nu_i - 1} |dy| \\
&= |\varepsilon|^s |\eta| \prod_{i=1}^{k} \int_{P^{j_i}} |y_i|^{N_i s + \nu_i - 1} |dy_i| \prod_{i=k+1}^{n} \int_{P^{j_i}} |dy_i| \\
&= |\varepsilon|^s |\eta| q^{-\sum_{i=k+1}^{n} j_i} \left(\frac{q-1}{q} \right)^k \prod_{i=1}^{k} \frac{q^{-j_i(N_i s + \nu_i)}}{1 - q^{-N_i s - \nu_i}}.
\end{aligned}$$

This shows that $Z_f(s)$ is a rational function of q^{-s}. Moreover, these calculations imply that the integral

$$\int_V |(f \circ g)(y)|^s |g^* dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid 0 \leq i \leq k\}$. Consequently, the integral

$$\int_X |f(x)|^s |dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid i \in T\}$. We obtain also from this calculation that every pole of $Z_f(s)$ is of the form

$$-\frac{\nu_i}{N_i} + \frac{2k\pi\sqrt{-1}}{N_i \log q},$$

with $k \in \mathbb{Z}$ and $i \in T$. These values are called the candidate poles of $Z_f(s)$. If $i \in T$ is fixed, the values $-\nu_i/N_i + (2k\pi\sqrt{-1})/(N_i \log q)$, $k \in \mathbb{Z}$, are called the candidate poles of $Z_f(s)$ associated to E_i.

Let s_0 be a candidate pole of $Z_f(s)$. Because the poles of $1/(1-q^{-N_i s - \nu_i})$ have order one, we define the expected order $m = m(s_0)$ of s_0 as the highest number of E_i's with candidate pole s_0 and with nonempty intersection. The order of s_0 is of course less than or equal to m. It is less than m if and only if b_{-m}, which is defined by the MacLaurin series

$$\frac{b_{-m}}{(s-s_0)^m} + \frac{b_{-m+1}}{(s-s_0)^{m-1}} + \cdots + b_0 + b_1(s-s_0) + \cdots$$

of $Z_f(s)$ at s_0, is equal to zero. Remark that a candidate pole of expected order one is a pole if and only if $b_{-1} \neq 0$.

(2.3) Fix a uniformizing parameter π for R. For $z \in K$ let $\mathrm{ac}\, z := z\pi^{-\mathrm{ord}\, z}$ be the angular component of z. Let χ be a character of $R^\times$, i.e., a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. The generalised Igusa's p-adic zeta function of f is now defined by

$$Z_{f,\chi}(s) = \int_X \chi(\mathrm{ac}\, f(x))|f(x)|^s \,|dx|$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) \geq 0$.

In the one variabele case we have now that

$$\int_{P^j} \chi(\mathrm{ac}\, x)|x|^{\alpha-1}\,|dx| = \begin{cases} \frac{q-1}{q}\frac{q^{-j\alpha}}{1-q^{-\alpha}} & \text{if } \chi = 1 \\ 0 & \text{if } \chi \neq 1 \end{cases}$$

for a complex number α with $\mathrm{Re}(\alpha) > 0$. The integral is not defined if $\mathrm{Re}(\alpha) \leq 0$.

Because $\chi(\mathrm{ac}\,\varepsilon)$ is a locally constant function on each chart, we can choose a finite set J of good charts (V, y) for $(f \circ g, g^*dx)$ such that $|\varepsilon|$, $|\eta|$ and $\chi(\mathrm{ac}\,\varepsilon)$ are constant on each chart, the V's form a partition of Y and for each chart (V, y) we have $y(V) = P^j$ for some $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$. Analogously as in (2.2) we obtain

$$\begin{aligned} Z_{f,\chi}(s) &= \sum_{(V,y)\in J} \int_V \chi(\mathrm{ac}\,(f\circ g)(y))|(f\circ g)(y)|^s \,|g^*dx| \\ &= \sum_{(V,y)\in J} \chi(\mathrm{ac}\,\varepsilon)|\varepsilon|^s|\eta|q^{-\sum_{i=k+1}^n j_i} \prod_{i=1}^k \int_{P^{j_i}} \chi^{N_i}(\mathrm{ac}\, y_i)|y_i|^{N_i s+\nu_i-1}\,|dy_i|. \end{aligned}$$

This implies that $Z_{f,\chi}(s)$ is again a rational function of q^{-s}, so that it extends to a meromorphic function $Z_{f,\chi}(s)$ on $\mathbb{C}$. The integral

$$\int_X \chi(\mathrm{ac}\, f(x))|f(x)|^s \,|dx|$$

is defined if and only if $\mathrm{Re}(s) > \max\{-\nu_i/N_i \mid i \in T\}$. We obtain that every pole of $Z_{f,\chi}(s)$ is of the form $-\nu_i/N_i + (2k\pi\sqrt{-1})/(N_i \log q)$, with $k \in \mathbb{Z}$ and $i \in T$ such that $\chi^{N_i} = 1$. These values are called the candidate poles of $Z_{f,\chi}(s)$. Obviously, we associate only candidate poles to E_i if $\chi^{N_i} = 1$. The expected order $m = m(s_0)$ of a candidate pole s_0 of $Z_{f,\chi}(s)$ is the highest number of E_i's with candidate pole s_0 (thus also with $\chi^{N_i} = 1$) and with nonempty intersection.

Remark that everything we have done up till now is well known. More details can be found for example in [Ig3].

(2.4) Let X be an open and compact subset of K^n. Let $\xi, f, f_1, \ldots, f_l$ be K-analytic functions on X. Let a_i, b_i, $1 \leq i \leq l$, be nonnegative integers. We associate to these data the zeta function

$$Z(s_1, \ldots, s_l) = \int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1+b_1} \ldots |f_l|^{a_l s_l+b_l}|dx|,$$

which is defined on a set U that contains all points $(s_1,\ldots,s_l)\in\mathbb{C}^l$ with $\mathrm{Re}(s_i)\geq -b_i/a_i$ if f_i vanishes on X and s_i arbitrary if f_i does not vanish on X. Loeser [Lo2] already studied this zeta function. By looking at an embedded resolution of $\xi f f_1\ldots f_l$, one proves analogously as in (2.2) that $Z(s_1,\ldots,s_l)$ is a rational function of $q^{-s_1},\ldots,q^{-s_l}$. Consequently, it extends to a meromorphic function on $\mathbb{C}^l$, which we also denote by $Z(s_1,\ldots,s_l)$. As before, we can also obtain an explicit description of U, which turns out to be an open subset of $\mathbb{C}^l$.

The meromorphic continuation of a function h will be denoted by $[h]^{\mathrm{mc}}$ and the evaluation of this meromorphic continuation in the point $s=s_0$ of the domain will be denoted by $[h]^{\mathrm{mc}}_{s=s_0}$.

In our study of Igusa's p-adic zeta function, we will have to deal with expressions of the form

$$\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s+b_1}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0}.$$

The zeta function in more complex variables can be used to modify this expression. If $U\cap\{(s_1,\ldots,s_l)\in\mathbb{C}^l\mid s_1=s_0\}\neq\emptyset$, then

$$\begin{aligned}
&\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s+b_1}|f_2|^{a_2 s+b_2}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0}\\
&=\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1+b_1}|f_2|^{a_2 s_2+b_2}\ldots|f_l|^{a_l s_l+b_l}|dx|\right]^{\mathrm{mc}}_{s_1=\cdots=s_l=s_0}\\
&=\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_0+b_1}|f_2|^{a_2 s+b_2}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}_{s=s_0}.
\end{aligned}$$

We explain the first equality. The composition of the map

$$A:\mathbb{C}\to\mathbb{C}^l: s\mapsto(s,\ldots,s)$$

with the meromorphic function $Z:\mathbb{C}^l\to\mathbb{C}$ which sends $(s_1,\ldots,s_l)\in U$ to

$$\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s_1+b_1}\ldots|f_l|^{a_l s_l+b_l}|dx|$$

is a meromorphic function on $\mathbb{C}$ which is equal to the meromorphic function

$$\left[\int_X \chi(\mathrm{ac}\, f)|\xi||f_1|^{a_1 s+b_1}\ldots|f_l|^{a_l s+b_l}|dx|\right]^{\mathrm{mc}}$$

because they agree on an open subset of $\mathbb{C}$. Consequently, the first equality is nothing more than $(Z\circ A)(s)=Z(A(s))$. For the second equality, we have to use the map

$$B:\mathbb{C}\to\mathbb{C}^l: s\mapsto(s_0,s,\ldots,s).$$

(2.5) Let f be a K-analytic function on an open and compact subset X of K^n and let $g : Y \to X$ be an embedded resolution of (f, dx) as in (2.1). For $I \subset T$ denote $E_I = \cap_{i\in I} E_i$. Let χ be a character of $R^\times$. Let s_0 be a candidate pole of $Z_{f,\chi}(s)$ and let m be its expected order. Let E_I, $I \in S$, be all the nonempty intersections of m varieties E_i, $i \in T$, with candidate pole s_0 (and thus also with $\chi^{N_i} = 1$). Fix $I \in S$ and suppose for the ease of notation that $I = \{1, \ldots, m\}$. Let W_1 and W_2 be open and compact subsets of Y which satisfy $E_I \cap W_1 = E_I \cap W_2 \neq \emptyset$ and which do not meet any E_K, $K \in S \setminus \{I\}$. Then the contribution of W_1 to b_{-m} and the contribution of W_2 to b_{-m} are the same because they are both equal to the contribution of $W_1 \cap W_2$ to b_{-m}. Consequently we can speak of the contribution of $E_I \cap W_1 = E_I \cap W_2$ to b_{-m}. In particular, the contribution of E_I to b_{-m} is well defined.

Consider a set J of disjoint compact charts (V, y) that intersect E_I, that cover E_I and that are disjoint with all E_K, $K \in S \setminus \{I\}$. This set J is necessarily finite and the contribution of E_I to b_{-m} is the sum over J of the contributions

$$\lim_{s\to s_0} (s - s_0)^m \left[\int_V \chi(\mathrm{ac}\,(f\circ g)(y)) |(f\circ g)(y)|^s |g^* dx| \right]^{\mathrm{mc}}$$

of V to b_{-m}.

We introduce some notation. Let (V, y) be a chart. We have that $\overline{y} = (y_{m+1}, \ldots, y_n)$ determines a chart on the closed submanifold $\overline{V}$ defined by $y_1 = \cdots = y_m = 0$. Denote $dy_{m+1} \wedge \cdots \wedge dy_n$ by $d\overline{y}$. It is a volume form on $\overline{V}$. If $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$, then we denote $P^{j_{m+1}} \times \cdots \times P^{j_n}$ by $\overline{P^j}$.

Suppose that (V, y) is a chart such that $E_1, \ldots, E_m$ have equations $y_1 = 0, \ldots, y_m = 0$ respectively, and such that

$$f\circ g = \alpha \prod_{i=1}^{m} y_i^{N_i} \qquad \text{and} \qquad g^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1} dy$$

on V, for K-analytic functions α and β on V with $|\alpha|$, $|\beta|$ and $\chi(\mathrm{ac}\,\alpha)$ *independent of* $y_1, \ldots, y_m$. Remark that a good chart (V, y) for $(f\circ g, g^* dx)$ in which $|\varepsilon|$, $|\eta|$ and $\chi(\mathrm{ac}\,\varepsilon)$ are constant satisfies this condition for $\alpha = \varepsilon \prod_{i=m+1}^{k} y_i^{N_i}$ and $\beta = \eta \prod_{i=m+1}^{k} y_i^{\nu_i - 1}$. Remark also that $\overline{V} = V \cap E_I$. Suppose also that $y(V)$ is of the form P^j with $j = (j_1, \ldots, j_n) \in (\mathbb{Z}_{\geq 0})^n$. Then

$$\begin{aligned}
&\lim_{s\to s_0} (s - s_0)^m \left[\int_V \chi(\mathrm{ac}\,(f\circ g)(y)) |(f\circ g)(y)|^s |g^* dx| \right]^{\mathrm{mc}} \\
&= \lim_{s\to s_0} (s - s_0)^m \left[\int_{P^j} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| \prod_{i=1}^{m} \chi^{N_i}(\mathrm{ac}\, y_i) |y_i|^{N_i s + \nu_i - 1} |dy| \right]^{\mathrm{mc}} \\
&= \left(\prod_{i=1}^{m} \lim_{s\to s_0} (s - s_0) \left[\int_{P^{j_i}} |y_i|^{N_i s + \nu_i - 1} |dy_i| \right]^{\mathrm{mc}} \right) \left[\int_{\overline{P^j}} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| |d\overline{y}| \right]^{\mathrm{mc}}_{s = s_0}
\end{aligned}$$

$$= \left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{\overline{V}} \chi(\operatorname{ac}\alpha)|\alpha|^s |\beta| |d\overline{y}|\right]^{\mathrm{mc}}_{s=s_0}$$

We have derived that the last expression is the contribution of V to b_{-m}. Consequently, the only aspect of the chart (V, y) it depends on is $\overline{V}$. In the next section we will see that we do not have to require that $|\alpha|$ and $|\beta|$ are independent of $y_1, \ldots, y_m$ and that we are in an embedded resolution to have this independence.

(2.6) Suppose that $g : Y = Y_t \to X = Y_0$ is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$. Suppose that each g_i is a blowing-up along a K-analytic closed submanifold of codimension bigger than one which has only normal crossings with the union of the exceptional varieties of $g_1 \circ \cdots \circ g_{i-1}$. Let $I = \{1, \ldots, m\} \in S$ as in (2.5). Let $r \in \{0, \ldots, t\}$. Suppose that E_I already exists in Y_r and that the E_i, $i \in I$, intersect transversally in Y_r. Remark that the last condition is satisfied if all the E_i, $i \in I$, are exceptional. We will write $E_I \subset Y_r$ if we want to stress that we consider E_I as a subset of Y_r.

We call a chart (V, y) a good chart for $E_I \subset Y_r$ if (V, y) is a chart on Y_r such that V intersects E_I and such that $y_1 = 0, \ldots, y_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on V.

Let (V, y) be a good chart for $E_I \subset Y_r$. Then we have

$$f \circ g_1 \circ \cdots \circ g_r = \alpha \prod_{i=1}^{m} y_i^{N_i} \text{ and } (g_1 \circ \cdots \circ g_r)^* dx = \beta \prod_{i=1}^{m} y_i^{\nu_i - 1} dy$$

on V, for K-analytic functions α and β on V.

We will now prove that the only aspect of the chart (V, y) that

$$\left[\int_{\overline{V}} \chi(\operatorname{ac}\alpha)|\alpha|^s |\beta| |d\overline{y}|\right]^{\mathrm{mc}}_{s=s_0} \qquad (1)$$

depends on is $\overline{V}$.

Let (W, z) be another chart on Y_r such that $\overline{V} = \overline{W}$ and such that $z_1 = 0, \ldots, z_m = 0$ are the equations of respectively $E_1, \ldots, E_m$ on W. We may suppose that $V = W$ because we can restrict them both to $V \cap W$. For every $i \in \{1, \ldots, m\}$ there exists a nonvanishing K-analytic function f_i on V such that $y_i = f_i z_i$ because y_i and z_i are equations of the same E_i. Thus

$$\begin{aligned} f \circ g_1 \circ \cdots \circ g_r &= \alpha \prod_{i=1}^{m} (f_i z_i)^{N_i} \\ &= \alpha \left(\prod_{i=1}^{m} f_i^{N_i}\right) \prod_{i=1}^{m} z_i^{N_i} \end{aligned}$$

and

$$
\begin{aligned}
(g_1 \circ \cdots \circ g_r)^* dx &= \beta \prod_{i=1}^{m} (f_i z_i)^{\nu_i - 1} \det\left(\frac{\partial y}{\partial z}\right) dz \\
&= \beta \left(\prod_{i=1}^{m} f_i^{\nu_i - 1}\right) \det\left(\frac{\partial y}{\partial z}\right) \prod_{i=1}^{m} z_i^{\nu_i - 1} dz.
\end{aligned}
$$

We have to prove that (1) is equal to

$$
\left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha)|\alpha|^s|\beta| \left(\prod_{i=1}^{m} \chi^{N_i}(\mathrm{ac}\, f_i)|f_i|^{N_i s + \nu_i - 1}\right) \left|\det\left(\frac{\partial y}{\partial z}\right)\right| |d\overline{z}|\right]^{\mathrm{mc}}_{s=s_0}. \qquad (2)
$$

In (1) we have that $|d\overline{y}| = |\det(\partial\overline{y}/\partial\overline{z})||d\overline{z}|$. Recall that $\chi^{N_i} = 1$ for $i \in \{1, \ldots, m\}$. Because f_i, $i \in I$, is a nonvanishing function, we may replace each $|f_i|^{N_i s + \nu_i - 1}$ in (2) by $|f_i|^{N_i s_0 + \nu_i - 1}$ according to (2.4), and this is equal to $|f_i|^{-1}$ because $N_i s_0 + \nu_i = 0$ for $i \in I$. Consequently, we have to prove that

$$
\begin{aligned}
&\left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha)|\alpha|^s|\beta| \left|\det\left(\frac{\partial \overline{y}}{\partial \overline{z}}\right)\right| |d\overline{z}|\right]^{\mathrm{mc}}_{s=s_0} \\
&= \left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha)|\alpha|^s|\beta| \prod_{i=1}^{m} |f_i|^{-1} \left|\det\left(\frac{\partial y}{\partial z}\right)\right| |d\overline{z}|\right]^{\mathrm{mc}}_{s=s_0}.
\end{aligned}
$$

Because $(\partial y/\partial z)$ is equal to

$$
\begin{pmatrix}
f_1 + z_1\frac{\partial f_1}{\partial z_1} & \cdots & z_1\frac{\partial f_1}{\partial z_m} & z_1\frac{\partial f_1}{\partial z_{m+1}} & \cdots & z_1\frac{\partial f_1}{\partial z_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_m\frac{\partial f_m}{\partial z_1} & \cdots & f_m + z_m\frac{\partial f_m}{\partial z_m} & z_m\frac{\partial f_m}{\partial z_{m+1}} & \cdots & z_m\frac{\partial f_m}{\partial z_n} \\
\frac{\partial y_{m+1}}{\partial z_1} & \cdots & \frac{\partial y_{m+1}}{\partial z_m} & \frac{\partial y_{m+1}}{\partial z_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial z_n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial y_n}{\partial z_1} & \cdots & \frac{\partial y_n}{\partial z_m} & \frac{\partial y_n}{\partial z_{m+1}} & \cdots & \frac{\partial y_n}{\partial z_n}
\end{pmatrix},
$$

we obtain that

$$
\left[\det\left(\frac{\partial y}{\partial z}\right)\right]_{z_1=\cdots=z_m=0} = \left[\left(\prod_{i=1}^{m} f_i\right) \det\left(\frac{\partial \overline{y}}{\partial \overline{z}}\right)\right]_{z_1=\cdots=z_m=0}.
$$

Consequently we have proved our statement.

(2.7) Definition-Proposition. *(We are still working in the situation and with the notation explained in the beginning of (2.6).) Suppose that $r < t$. For an* *open and compact subset U of $E_I \subset Y_r$ which is equal to $\overline{V}$ for some good chart (V, y) for $E_I \subset Y_r$, we define the contribution of U to b_{-m} as*

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{\overline{V}} \chi(\operatorname{ac} \alpha)|\alpha|^s |\beta||d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}.$$

For an arbitrary open and compact subset U of $E_I \subset Y_r$, we first take a partition consisting of subsets for which the contribution is already defined above, and we define the contribution of U to b_{-m} as the sum of these contributions. Remark that this definition is consistent with our result for $r = t$ in (2.5).
The result is that this terminology is appropriate in the sense that the contribution of an open and compact subset U of $E_I \subset Y_r$ to b_{-m} is equal to the contribution of the strict transform of U under $g_{r+1} \circ \cdots \circ g_t$ to b_{-m}.

Proof. Let U be an open and compact subset of $E_I \subset Y_r$. We will now prove that the contribution of U to b_{-m} is equal to the contribution of the strict transform of U under $g_{r+1} \circ \cdots \circ g_t$ to b_{-m}.

Because $g_{r+1} \circ \cdots \circ g_t$ is a composition of a finite number of blowing-ups, it is enough to prove that the contribution of U to b_{-m} is equal to the contribution of $\widetilde{U}$, the strict transform of U under g_{r+1}, to b_{-m}.

We may suppose that U is of the form $\overline{V}$ for some good chart (V, y) for $E_I \subset Y_r$. The contribution of U to b_{-m} is then equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{\widetilde{U}} \chi(\operatorname{ac} g_{r+1}^* \alpha)|g_{r+1}^* \alpha|^s |g_{r+1}^* \beta||g_{r+1}^* d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}.$$

Remark that the definition of the contribution of $\widetilde{U}$ to b_{-m} uses also an integral over $\widetilde{U}$.

Because the centre of g_{r+1} does not contain E_I and has only normal crossings with $E_1 \cup \cdots \cup E_m$, we may suppose that g_{r+1} is the blowing-up along $y_1 = \cdots = y_a = y_{m+1} = \cdots = y_b = 0$, where $0 \leq a \leq m < b \leq n$. The transformation

$$(z_1, \ldots, z_n) \mapsto (z_1 z_b, \ldots, z_a z_b, z_{a+1}, \ldots, z_m, z_{m+1} z_b, \ldots, z_{b-1} z_b, z_b, z_{b+1}, \ldots, z_n)$$

determines coordinates z on an open subset O of Y_{r+1}. If we permute $y_{m+1}, \ldots, y_b$, it represents another transformation which determines coordinates on another open subset of Y_{r+1}; the open subsets we obtain in this way cover $\widetilde{U}$.

Let W be an open and compact subset of O that intersects $E_I \subset Y_{r+1}$. Then (W, z) is a good chart for $E_I \subset Y_{r+1}$. It is enough to prove that the contribution of W to b_{-m} is equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q}\right) \left[\int_{\overline{W}} \chi(\operatorname{ac} g_{r+1}^* \alpha)|g_{r+1}^* \alpha|^s |g_{r+1}^* \beta||g_{r+1}^* d\overline{y}|\right]_{s=s_0}^{\mathrm{mc}}.$$

Because

$$f \circ g_1 \circ \cdots \circ g_{r+1} = g_{r+1}^* \alpha \left(\prod_{i=1}^{m} z_i^{N_i} \right) z_b^{\sum_{i=1}^{a} N_i}$$

and

$$\begin{aligned}(g_1 \circ \cdots \circ g_{r+1})^* dx &= g_{r+1}^* \beta \left(\prod_{i=1}^{m} z_i^{\nu_i - 1} \right) z_b^{\sum_{i=1}^{a} (\nu_i - 1)} z_b^{b-m-1+a} dz \\ &= g_{r+1}^* \beta \left(\prod_{i=1}^{m} z_i^{\nu_i - 1} \right) z_b^{\sum_{i=1}^{a} \nu_i} z_b^{b-m-1} dz,\end{aligned}$$

we obtain that the contribution of W to b_{-m} is equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q} \right) \left[\int_{\overline{W}} \chi(\mathrm{ac}\, g_{r+1}^* \alpha) |g_{r+1}^* \alpha|^s |g_{r+1}^* \beta| |z_b|^{\sum_{i=1}^{a} (N_i s + \nu_i)} |z_b|^{b-m-1} |d\overline{z}| \right]_{s=s_0}^{\mathrm{mc}}.$$

We can use (2.4) to simplify this expression because we know that $N_i s_0 + \nu_i = 0$ for $i \in \{1, \ldots, a\}$ and that $b - m - 1 \geq 0$. Because moreover $g_{r+1}^* d\overline{y} = z_b^{b-m-1} d\overline{z}$, we obtain that the contribution of W to b_{-m} is equal to

$$\left(\prod_{i=1}^{m} \frac{q-1}{qN_i \log q} \right) \left[\int_{\overline{W}} \chi(\mathrm{ac}\, g_{r+1}^* \alpha) |g_{r+1}^* \alpha|^s |g_{r+1}^* \beta| |g_{r+1}^* d\overline{y}| \right]_{s=s_0}^{\mathrm{mc}}.$$

This had to be proved. □

Remark. (i) The contribution of E_I to b_{-m} is not necessarily equal to the contribution of a 'very small' neighbourhood of $E_I \subset Y_r$ to b_{-m}, because it can happen that an E_K, $K \in S \setminus \{I\}$, lies above $E_I \subset Y_r$.
(ii) For the ease of notation we will sometimes neglect the first factor of this formula, which is not equal to zero.

(2.8) Let T_t be the set of all $j \in T \setminus I$ for which E_j intersects E_I in Y_t. Let F_j, $j \in T_t$, be the intersection of E_j and E_I in Y_t. We have that F_j has codimension one in $E_I \subset Y_t$. The set of all j, $j \in T_t$, for which $(g_{r+1} \circ \cdots \circ g_t)(F_j)$ has also codimension one in $E_I \subset Y_r$ will be denoted by T_r. For $j \in T_r$ we denote $(g_{r+1} \circ \cdots \circ g_t)(F_j)$ also by F_j and we put $\alpha_j = N_j s_0 + \nu_j$.

Let (V, y) be a good chart for E_I on Y_r on which F_j, $j \in T_r$, is given by $y_1 = \cdots = y_m = y_{m+1} = 0$. Write

$$\begin{aligned}\alpha(0, \ldots, 0, y_{m+1}, \ldots, y_n) &= y_{m+1}^{N_{j,r}} h_1 \qquad \text{and} \\ \beta(0, \ldots, 0, y_{m+1}, \ldots, y_n) &= y_{m+1}^{\nu_{j,r} - 1} h_2\end{aligned}$$

with h_1 and h_2 not divisible by y_{m+1}. Then we denote $N_{j,r} s_0 + \nu_{j,r}$ by $\alpha_{j,r}$.

We deduce now the relations that will be used later. In this paragraph we suppose that $m=1$ and that $E_I = E_r$ is created by the blowing-up at a point P of Y_{r-1}. Suppose that there exists a chart (V, y) centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a homogeneous polynomial for which every irreducible factor over $K^{\mathrm{alg\,cl}}$ is defined over K and for which the zero locus in $\mathbb{P}^{n-1}$ of every irreducible factor (over $K^{\mathrm{alg\,cl}}$) contains a nonsingular point defined over K. Remark that these conditions are satisfied if the lowest degree part is a product of linear factors defined over K. Write $f \circ g_1 \circ \cdots \circ g_{r-1} = e\left(\prod_{j \in T_r} f_j^{N_{j,r}}\right) + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho\left(\prod_{j \in T_r} f_j^{\nu_{j,r}-1}\right) dy$, where f_j is the equation of $F_j \subset E_r$ in the homogeneous coordinates $(y_1 : \cdots : y_n)$ on $E_r \subset Y_r$, $e \in K^\times$, θ is a power series with multiplicity larger than the degree of the homogeneous polynomial $\prod_{j \in T_j} f_j^{N_{j,r}}$ and ρ is a K-analytic function which does not vanish at P. Because the multiplicity of $f \circ g_1 \circ \cdots \circ g_{r-1}$ at P is equal to N_r, we obtain the first relation:

$$\sum_{j \in T_r} (\deg F_j) N_{j,r} = N_r. \qquad \text{(Relation 1)}$$

Our second relation will involve the $\alpha_{j,r}$, $j \in T_r$. There will appear differential forms with rational exponents in the calculations. One can make sense to this by considering them as an element of a tensor power of the module of rational differential forms (see [Ja]), but we will not give details here. Let $i \in \{1, \ldots, n\}$. We look at the chart $(O, z = (z_1, \ldots, z_n))$ on Y_r for which $g_r(z_1, \ldots, z_n) = (z_1 z_i, \ldots, z_{i-1} z_i, z_i, z_{i+1} z_i, \ldots, z_n z_i)$. Then

$$f \circ g_1 \circ \cdots \circ g_r = z_i^{N_r} \left(e \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{N_{j,r}} + z_i \frac{\theta \circ g_r}{z_i^{N_r+1}} \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx = z_i^{\nu_r - 1} (\rho \circ g_r) \left(\prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\nu_{j,r}-1} \right) dz.$$

Consequently the Poincaré residue of $(f \circ g_1 \circ \cdots \circ g_r)^{-\nu_r/N_r} (g_1 \circ \cdots g_r)^* dx$ on $E_I \subset Y_r$ (see [Ja]) is equal to

$$e^{-\nu_r/N_r} \rho(P) \prod_{j \in T_r} f_j(z_1, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n)^{\alpha_{j,r}-1} d\overline{z},$$

so that the canonical divisor of E_r is $\sum_{j \in T_r} (\alpha_{j,r} - 1) F_j$. Because we know that the degree of the canonical divisor on $E_r \cong \mathbb{P}^{n-1}$ is $-n$, we obtain the second relation:

$$\sum_{j \in T_r} (\deg F_j)(\alpha_{j,r} - 1) = -n. \qquad \text{(Relation 2)}$$

Remark that the condition on the lowest degree part of $f \circ g_1 \circ \cdots \circ g_{r-1}$ has to be satisfied because otherwise some terms on the left hand side are missing. We need the two relations which we just derived in section 3. In the next paragraph we will deduce that $\alpha_{j,r} = \alpha_j$ and that $N_{j,r} \equiv N_j \pmod{N_r}$ so that we obtain

$$\sum_{j \in T_r} (\deg F_j) N_j \equiv 0 \pmod{N_r} \qquad \text{and}$$
$$\sum_{j \in T_r} (\deg F_j)(\alpha_j - 1) = -n.$$

One can find these relations in a more general form in [Ve1], [Ve2] and [Ve4].

We prove that $\alpha_{j,r} = \alpha_j$ for $j \in T_r$. Because $g_{r+1} \circ \cdots \circ g_t$ is a composition of a finite number of blowing-ups, it is enough to prove that $\alpha_{j,r} = \alpha_{j,r+1}$. If the centre of g_{r+1} does not contain F_j, then $N_{j,r} = N_{j,r+1}$ and $\nu_{j,r} = \nu_{j,r+1}$ so that we are done. If the centre of g_{r+1} contains F_j, we may suppose that g_{r+1} is the blowing-up along $y_1 = \cdots = y_a = y_{m+1} = 0$, where $0 < a \leq m$. The relevant chart is determined by the transformation

$$(z_1, \ldots, z_n) \mapsto (z_1 z_{m+1}, \ldots, z_a z_{m+1}, z_{a+1}, \ldots, z_m, z_{m+1}, \ldots, z_n).$$

Because

$$f \circ g_1 \circ \cdots \circ g_{r+1} = g_{r+1}^* \alpha \left(\prod_{i=1}^{m} z_i^{N_i} \right) z_{m+1}^{\sum_{i=1}^{a} N_i}$$

and

$$(g_1 \circ \cdots \circ g_{r+1})^* dx = g_{r+1}^* \beta \left(\prod_{i=1}^{m} z_i^{\nu_i - 1} \right) z_{m+1}^{\sum_{i=1}^{a} \nu_i} dz,$$

we have to prove that $N_{j,r} s_0 + \nu_{j,r} = (N_{j,r} + \sum_{i=1}^{a} N_i) s_0 + (\nu_{j,r} + \sum_{i=1}^{a} \nu_i)$. This follows from the fact that $N_i s_0 + \nu_i = 0$ for $i \in \{1, \ldots, a\}$. Remark that it follows also from these calculations that

$$N_{j,r} \equiv N_j \bmod \gcd(N_1, \ldots, N_m).$$

(2.9) *Example.* We give an illustration which is easy and well known. Let $f = x_1^2 + x_2^2$. Let $X = \mathbb{Z}_p \times \mathbb{Z}_p$. We want to determine the poles of Igusa's p-adic zeta function associated to f. Notice that -1 is a square in $\mathbb{Q}_p$ if and only if -1 is a square $\mathbb{Z}/(p)$ and $p \neq 2$.

If -1 is a square in $\mathbb{Q}_p$, then (f, dx) has already normal crossings. We obtain a good chart for (f, dx) by applying the coordinate transformation $(y_1, y_2) \mapsto ((y_1 + y_2)/2, (y_1 - y_2)/(2a))$, where a denotes a square rooth of -1. Because $|a| = 1$ we obtain

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} |x_1^2 + x_2^2|^s |dx_1 \wedge dx_2| = \int_{\mathbb{Z}_p \times \mathbb{Z}_p} |y_1 y_2|^s |dy_1 \wedge dy_2|.$$

Consequently, the only candidate poles of $Z_f(s)$ are $-1 + (2k\pi\sqrt{-1})/(\log p)$, $k \in \mathbb{Z}$. They are all poles because $b_{-2} = ((p-1)/(p\log p))^2$ for each candidate pole.

If -1 is not a square in $\mathbb{Q}_p$, then (f, dx) has not normal crossings at the origin. We obtain an embedded resolution after one blowing-up g. Remark that the zero locus of f contains only the origin and that the zero locus of $f \circ g$ is equal to the exceptional curve E of g. We will use the two charts on the blowing-up determined by $(y_1, y_2) \mapsto (y_1y_2, y_2)$ and $(z_1, z_2) \mapsto (z_1, z_1z_2)$. The sets $\{(y_1, y_2) \mid y_1 \in \mathbb{Z}_p\,,\, y_2 = 0\}$ and $\{(z_1, z_2) \mid z_1 = 0\,,\, z_2 \in p\mathbb{Z}_p\}$ form a partition of E. The candidate poles of $Z_f(s)$ are $s_k = -1 + (2k\pi\sqrt{-1})/(2\log p)$, $k \in \mathbb{Z}$, and each b_{-1} is equal to

$$\left(\frac{p-1}{2p\log p}\right)\left(\left[\int_{\mathbb{Z}_p} |y_1^2+1|^s |dy_1|\right]^{\mathrm{mc}}_{s=s_k} + \left[\int_{p\mathbb{Z}_p} |1+z_2^2|^s |dz_2|\right]^{\mathrm{mc}}_{s=s_k}\right).$$

If $p \neq 2$, we have that $|1+x^2| = 1$ for every $x \in \mathbb{Z}_p$, so that $b_{-1} = (p^2 - 1)/(2p^2\log p)$. If $p = 2$, we have that $|1+x^2| = 1$ for every $x \in 2\mathbb{Z}_2$ and $|1+x^2| = 1/2$ for every $x \in 1 + 2\mathbb{Z}_2$, so that $b_{-1} = 1/(2\log 2)$ if k is even and $b_{-1} = 0$ if k is odd.

Remark that Igusa's p-adic zeta function of $x_1^2 + x_2^2$ can be calculated completely elementarily in all the cases.

3 The vanishing results

3.1 Curves

Let X be an open and compact subset of K^2. Let f be a K-analytic function on X. Let $g : Y \to X$ be an embedded resolution of f. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. Let χ be a character of $R^\times$.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part a (nonconstant) monomial. Then the contribution of E_r to the residue b_{-1} of $Z_{f,\chi}(s)$ at s_0 is zero.*

Remark. This proposition is essentially well known. Our proof differs slightly from the ones in [Ig2] and [Lo1] because we will calculate the contribution of E_r to b_{-1} just after the creation of E_r instead of on the embedded resolution. We incorporate this proof here because the same technique will be used in the proof of the more difficult result of section 3.2.

Proof. We may suppose that (V, y) is a chart centred at P such that $f \circ g_1 \circ \cdots \circ g_{r-1} = ey_1^k y_2^l + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{c-1} y_2^{d-1} dy$ with $k, l \in \mathbb{Z}_{\geq 0}$, $c, d \in \mathbb{Z}_{>0}$, $e \in K^\times$ and ρ, θ K-analytic functions satisfying $\rho(0,0) \neq 0$ and $\mathrm{mult}(\theta) > k + l$. We consider here the case that k and l are both not zero. The case that k or l is zero can be treated analogously.

We look at the chart $(O, z = (z_1, z_2))$ on Y_r for which $g_r(z_1, z_2) = (z_1, z_1 z_2)$. Then

$$\begin{aligned} f \circ g_1 \circ \cdots \circ g_r &= z_1^{k+l}\left(e z_2^l + z_1 \frac{\theta(z_1, z_1 z_2)}{z_1^{k+l+1}}\right) \qquad \text{and} \\ (g_1 \circ \cdots \circ g_r)^* dx &= \rho(z_1, z_1 z_2) z_1^{c+d-1} z_2^{d-1} dz. \end{aligned}$$

Remark that the equation of E_r is $z_1 = 0$, that $N_r = k + l$ and that $\nu_r = c + d$. Using the notation of (2.8), let $T_r = \{1, 2\}$ and let F_1 be the origin of this chart. The contribution to b_{-1} of an open and compact subset A of E_r which is contained in O is equal to

$$\left(\frac{q-1}{qN_r \log q}\right)\left[\int_A \chi(\mathrm{ac}\, e)\chi^l(\mathrm{ac}\, z_2)|e|^s|\rho(0,0)||z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}}.$$

Let $(O', z' = (z_1', z_2'))$ be the chart on Y_r for which $g_r(z_1', z_2') = (z_1' z_2', z_2')$. The origin of this chart is the point F_2. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset B of E_r which is contained in O' is equal to

$$\left(\frac{q-1}{qN_r \log q}\right)\left[\int_B \chi(\mathrm{ac}\, e)\chi^k(\mathrm{ac}\, z_1')|e|^s|\rho(0,0)||z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}}.$$

Because $\chi^{N_r} = 1$ (otherwise there are no candidate poles associated to E_r) and because $k + l = N_r$, we have that $\chi^k = 1$ if and only if $\chi^l = 1$.

Case 1: $\chi^k = \chi^l = 1$. Then the contribution of E_r to b_{-1} is equal to

$$\begin{aligned} &\left(\frac{\chi(\mathrm{ac}\, e)|e|^{s_0}|\rho(0,0)|(q-1)}{qN_r \log q}\right)\left(\left[\int_R |z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}} + \left[\int_P |z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}}\right) \\ = &\left(\frac{\chi(\mathrm{ac}\, e)|e|^{s_0}|\rho(0,0)|(q-1)}{qN_r \log q}\right)\left(\frac{q-1}{q}\frac{1}{1-q^{-\alpha_1}} + \frac{q-1}{q}\frac{q^{-\alpha_2}}{1-q^{-\alpha_2}}\right) \\ = &\left(\frac{\chi(\mathrm{ac}\, e)|e|^{s_0}|\rho(0,0)|(q-1)}{qN_r \log q}\right)\left(\frac{q-1}{q}\right)\left(\frac{1-q^{-\alpha_2}+q^{-\alpha_2}-q^{-\alpha_1-\alpha_2}}{(1-q^{-\alpha_1})(1-q^{-\alpha_2})}\right) \\ = &\ 0. \end{aligned}$$

The last equality follows from $\alpha_1 + \alpha_2 = 0$, which is relation 2 of (2.8).

Case 2: $\chi^k \neq 1$ and $\chi^l \neq 1$. Then the contribution of E_r to b_{-1} is equal to zero because both terms in the sum

$$\left[\int_R \chi^l(\mathrm{ac}\, z_2)|z_2|^{ls+d-1}|dz_2|\right]_{s=s_0}^{\mathrm{mc}} + \left[\int_P \chi^k(\mathrm{ac}\, z_1')|z_1'|^{ks+c-1}|dz_1'|\right]_{s=s_0}^{\mathrm{mc}}$$

are equal to zero. □

3.2 Surfaces

Let X be an open and compact subset of K^3. Let f be a K-analytic function on X. Let $g : Y = Y_t \to X = Y_0$ be an embedded resolution of f which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$ with centre a K-analytic closed submanifold which has only normal crossings with the union of the exceptional surfaces in Y_{i-1} and with exceptional surface E_i.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2, y_3))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part of the form $ey_1^k y_2^l y_3^m (y_1 + y_2)^n$, with $e \in K^\times$ and $k, l, m, n \in \mathbb{Z}_{\geq 0}$. Then the contribution of E_r to the residue b_{-1} of $Z_f(s)$ at s_0 is zero.*

Proof. We may suppose that $f \circ g_1 \circ \cdots \circ g_{r-1} = ey_1^k y_2^l y_3^m (y_1 + y_2)^n + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{a-1} y_2^{b-1} y_3^{c-1} (y_1 + y_2)^{d-1} dy$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and ρ, θ K-analytic functions satisfying $\rho(0,0) \neq 0$ and $\text{mult}(\theta) > k+l+m+n$. Remark that at least one of the numbers a, b, d is equal to 1. We consider here the case that k, l, m and n are all different from zero. The other cases are treated analogously. Let $T_r = \{1, 2, 3, 4\}$ and suppose that F_i, $i \in \{1, 2, 3\}$, is given by $y_i = 0$ and that F_4 is given by $y_1 + y_2 = 0$ in the homogeneous coordinates $(y_1 : y_2 : y_3)$ on $E_r \subset Y_r$.

We look at the chart $(O, z = (z_1, z_2, z_3))$ on Y_r for which $g_r(z_1, z_2, z_3) = (z_1 z_3, z_2 z_3, z_3)$. Then

$$\begin{aligned} f \circ g_1 \circ \cdots \circ g_r &= z_3^{k+l+m+n} \left(e z_1^k z_2^l (z_1 + z_2)^n + z_3 \frac{\theta(z_1 z_3, z_2 z_3, z_3)}{z_3^{k+l+m+n+1}} \right) \qquad \text{and} \\ (g_1 \circ \cdots \circ g_r)^* dx &= \rho(z_1 z_3, z_2 z_3, z_3) z_1^{a-1} z_2^{b-1} z_3^{a+b+c+d-2} (z_1 + z_2)^{d-1} dz. \end{aligned}$$

Remark that the equation of E_r is $z_3 = 0$, that $N_r = k + l + m + n$ and that $\nu_r = a + b + c + d - 1$. The contribution to b_{-1} of an open and compact subset A of E_r which is contained in O is equal to

$$\left(\frac{q-1}{qN_r \log q} \right) \left[\int_A |e|^s |\rho(0,0,0)| |z_1|^{ks+a-1} |z_2|^{ls+b-1} |z_1 + z_2|^{ns+d-1} |dz_1 \wedge dz_2| \right]_{s=s_0}^{\text{mc}}.$$

Let $(O', z' = (z'_1, z'_2, z'_3))$ be the chart on Y_r for which $g_r(z'_1, z'_2, z'_3) = (z'_1 z'_2, z'_2, z'_2 z'_3)$. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset B of E_r which is contained in O' is equal to

$$\left(\frac{q-1}{qN_r \log q} \right) \left[\int_B |e|^s |\rho(0,0,0)| |z'_1|^{ks+a-1} |z'_3|^{ms+c-1} |z'_1 + 1|^{ns+d-1} |dz'_1 \wedge dz'_3| \right]_{s=s_0}^{\text{mc}}.$$

Let $(O'', z'' = (z''_1, z''_2, z''_3))$ be the chart on Y_r for which $g_r(z''_1, z''_2, z''_3) = (z''_1, z''_1 z''_2, z''_1 z''_3)$. Analogously as before, we obtain that the contribution to b_{-1} of an open and compact subset C of E_r which is contained in O'' is equal to

$$\left(\frac{q-1}{qN_r\log q}\right)\left[\int_C |e|^s|\rho(0,0,0)||z_2''|^{ls+b-1}|z_3''|^{ms+c-1}|1+z_2''|^{ns+d-1}|dz_2''\wedge dz_3''|\right]^{\mathrm{mc}}_{s=s_0}.$$

Now we take $A = P\times P$, $B = P\times R$ and $C = R\times R$. Because these sets form a partition of E_r, the contribution to b_{-1} of E_r is the sum of the three contributions above.

We have to prove that the contribution to b_{-1} of E_r is equal to zero, so we have to prove that

$$\begin{aligned}
&\left[\int_A |z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_1\wedge dz_2|\right]^{\mathrm{mc}}_{s=s_0}\\
&+\left[\int_B |z_1'|^{ks+a-1}|z_3'|^{ms+c-1}|z_1'+1|^{ns+d-1}|dz_1'\wedge dz_3'|\right]^{\mathrm{mc}}_{s=s_0} \qquad (*)\\
&+\left[\int_C |z_2''|^{ls+b-1}|z_3''|^{ms+c-1}|1+z_2''|^{ns+d-1}|dz_2''\wedge dz_3''|\right]^{\mathrm{mc}}_{s=s_0}
\end{aligned}$$

is equal to zero.

To calculate the first term in $(*)$, we partition A into

$$\begin{aligned}
A_1 &= \{(z_1,z_2)\in P\times P \mid \mathrm{ord}\, z_1 > \mathrm{ord}\, z_2\} = \bigsqcup_{i\in\mathbb{Z}_{>0}} \{(z_1,z_2)\mid \mathrm{ord}\, z_1 > \mathrm{ord}\, z_2 = i\}\\
A_2 &= \{(z_1,z_2)\in P\times P \mid \mathrm{ord}\, z_1 < \mathrm{ord}\, z_2\} = \bigsqcup_{i\in\mathbb{Z}_{>0}} \{(z_1,z_2)\mid i = \mathrm{ord}\, z_1 < \mathrm{ord}\, z_2\}\\
A_3 &= \{(z_1,z_2)\in P\times P \mid \mathrm{ord}\, z_1 = \mathrm{ord}\, z_2\} = \bigsqcup_{i\in\mathbb{Z}_{>0}} \{(z_1,z_2)\mid \mathrm{ord}\, z_1 = \mathrm{ord}\, z_2 = i\}
\end{aligned}$$

The contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\left[\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}|z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_2|\right)|dz_1|\right]^{\mathrm{mc}}_{s=s_0}\\
&= \left[\sum_{i=1}^{\infty}\frac{q-1}{q}q^{-i}q^{-i(ls+b-1)}q^{-i(ns+d-1)}\int_{P^{i+1}}|z_1|^{ks+a-1}|dz_1|\right]^{\mathrm{mc}}_{s=s_0}\\
&= \left[\sum_{i=1}^{\infty}\frac{q-1}{q}q^{-i}q^{-i(ls+b-1)}q^{-i(ns+d-1)}\frac{q-1}{q}\frac{q^{-i(ks+a)}}{q^{ks+a}-1}\right]^{\mathrm{mc}}_{s=s_0}\\
&= \left[\left(\frac{q-1}{q}\right)^2\frac{1}{q^{ks+a}-1}\sum_{i=1}^{\infty}q^{-i(ks+a+ls+b+ns+d-1)}\right]^{\mathrm{mc}}_{s=s_0}\\
&= \left[\left(\frac{q-1}{q}\right)^2\frac{1}{(q^{ks+a}-1)(q^{ks+a+ls+b+ns+d-1}-1)}\right]^{\mathrm{mc}}_{s=s_0}
\end{aligned}$$

$$= \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_1}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}. \tag{3}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to

$$\left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_2}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}. \tag{4}$$

The contribution of A_3 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\left[\sum_{i=1}^{\infty} \int_{(P^i\setminus P^{i+1})^2} |z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_1\wedge dz_2|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\sum_{i=1}^{\infty} q^{-i(ks+a-1)}q^{-i(ls+b-1)} \int_{(P^i\setminus P^{i+1})^2} |z_1+z_2|^{ns+d-1}|dz_1\wedge dz_2|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \Bigg[\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b-2)} \int_{P^i\setminus P^{i+1}} \Bigg(\int_{-z_2+P^{i+1}} |z_1+z_2|^{ns+d-1}|dz_1| \\
&\qquad + \int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})} |z_1+z_2|^{ns+d-1}|dz_1|\Bigg)|dz_2|\Bigg]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b-2)} \int_{P^i\setminus P^{i+1}} \frac{q-1}{q}\frac{q^{-i(ns+d)}}{q^{ns+d}-1} + \frac{q-2}{q}q^{-i}q^{-i(ns+d-1)}|dz_2|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b-2)}\frac{q-1}{q}q^{-i}\left(\frac{q-1}{q}\frac{q^{-i(ns+d)}}{q^{ns+d}-1} + \frac{q-2}{q}q^{-i(ns+d)}\right)\right]_{s=s_0}^{\mathrm{mc}} \\
&= \Bigg[\left(\frac{q-1}{q}\right)^2 \frac{1}{q^{ns+d}-1}\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b+ns+d-1)} \\
&\qquad + \frac{q-1}{q}\frac{q-2}{q}\sum_{i=1}^{\infty} q^{-i(ks+a+ls+b+ns+d-1)}\Bigg]_{s=s_0}^{\mathrm{mc}}
\end{aligned}$$

$$= \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_4}-1)(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)} \tag{5}$$

$$+ \left(\frac{q-1}{q}\right)\left(\frac{q-2}{q}\right)\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}. \tag{6}$$

The second term of $(*)$ is equal to

$$\begin{aligned}
&\left[\int_P |z_1|^{ks+a-1}|z_1+1|^{ns+d-1}|dz_1| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\int_P |z_1|^{ks+a-1}|dz_1| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}}
\end{aligned}$$

$$= \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_1}-1)(1-q^{-\alpha_3})}. \tag{7}$$

The third term of $(*)$ is equal to

$$\left[\int_R |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| \int_R |z_3|^{ms+c-1}|dz_3|\right]_{s=s_0}^{\mathrm{mc}}$$

$$\begin{aligned}
&= \Bigg[\Bigg(\int_{R\setminus(P\cup -1+P)} |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| + \int_P |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2| \\
&\quad + \int_{-1+P} |z_2|^{ls+b-1}|1+z_2|^{ns+d-1}|dz_2|\Bigg)\left(\int_R |z_3|^{ms+c-1}|dz_3|\right)\Bigg]_{s=s_0}^{\mathrm{mc}} \\
&= \left[\left(1-\frac{2}{q}+\int_P |z_2|^{ls+b-1}|dz_2| + \int_{-1+P}|1+z_2|^{ns+d-1}|dz_2|\right)\left(\int_R |z_3|^{ms+c-1}|dz_3|\right)\right]_{s=s_0}^{\mathrm{mc}} \\
&= \left(1-\frac{2}{q}+\frac{q-1}{q}\frac{1}{q^{\alpha_2}-1}+\frac{q-1}{q}\frac{1}{q^{\alpha_4}-1}\right)\left(\frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}}\right) \\
&= \left(\frac{q-1}{q}\right)\left(\frac{q-2}{q}\right)\frac{1}{1-q^{-\alpha_3}} && (8) \\
&\quad + \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_2}-1)(1-q^{-\alpha_3})} && (9) \\
&\quad + \left(\frac{q-1}{q}\right)^2 \frac{1}{(q^{\alpha_4}-1)(1-q^{-\alpha_3})}. && (10)
\end{aligned}$$

Relation 2 of (2.8) is $\alpha_1+\alpha_2+\alpha_3+\alpha_4-1=0$, so that we obtain

$$\begin{aligned}
\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}+\frac{1}{1-q^{-\alpha_3}} &= \frac{1-q^{-\alpha_3}+q^{\alpha_1+\alpha_2+\alpha_4-1}-1}{(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)(1-q^{-\alpha_3})} \\
&= \frac{q^{\alpha_1+\alpha_2+\alpha_3+\alpha_4-1}-1}{(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)(q^{\alpha_3}-1)} \\
&= 0,
\end{aligned}$$

and consequently (3) + (7) = 0. Analogously, we obtain that (4) + (9) = (5) + (10) = (6) + (8) = 0. Consequently, the contribution of E_r to b_{-1} is equal to zero. □

Remark. (i) If k, l, m and n are not all different from zero, then the same calculations are valid. Now, $T_r \subset \{1,2,3,4\}$ and F_i, $i \in T_r$, is given by the same equation as before. If $i \in \{1,2,3,4\} \setminus T_r$, we have to put formally $\alpha_i = 1$.
(ii) Suppose that we are in the same situation as in the proposition and let χ be an arbitrary character of $R^\times$. Then one can show that the contribution of E_r to the residue b_{-1} of $Z_{f,\chi}(s)$ at s_0 is zero. The proof consists of very long calculations involving character sums. This will appear elsewhere.

4 Determination of the smallest poles

The main ideas and results of this section have the same flavour as those in [SV], where the local topological zeta function is studied. However it is here sometimes more complicated because the field K is not algebraically closed.

4.1 Curves

(4.1.1) In this section we will determine $\mathcal{P}_2^K \cap]-\infty, -1/2[$. Let f be a K-analytic function on an open and compact subset of K^2 and let g be the *minimal* embedded resolution of f. The poles of $Z_f(s)$ with real part less than $-1/2$ and different from -1 are only associated to exceptional curves. Consequently, these poles are completely determined by the germs of f at the points where f has not normal crossings. It is thus sufficient to study the germs of K-analytic functions at the origin, which will be identified with the convergent power series. The set of all convergent power series in the variables x and y is classically denoted by $K\ll x, y\gg$.

(4.1.2) Let $f \in K\ll x, y\gg$. Let $g : Y \to X$ be the *minimal* embedded resolution of a representative of f. Write $g = g_1 \circ \cdots \circ g_t : Y = Y_t \to X = Y_0$ as a composition of blowing-ups $g_i : Y_i \to Y_{i-1}$, $i \in \{1, \ldots, t\}$. The exceptional curve of g_i and also the strict transforms of this curve are denoted by E_i. Let T be as in (2.1) and obviously we suppose that $\{1, \ldots, t\} \subset T$.

Let $k \in \{1, \ldots, t\}$. Let $P \in Y_k$ be a point on an exceptional curve, i.e., a point which is mapped to the origin under the map $g_1 \circ \cdots \circ g_k$. The strict transform of f around P is defined as the germ at P of the K-analytic function $f \circ g_1 \circ \cdots \circ g_k$ devided by the highest possible powers of local equations of exceptional curves through P. Remark that the strict transform of f around P is defined modulo the germ of a K-analytic function which does not vanish at P as a factor.

We call a complex number 'a candidate pole of $Z_f(s)$' if it is a candidate pole associated to an E_i, $i \in T$, satisfying $0 \in g(E_i)$. A candidate pole of $Z_f(s)$ is called a pole of $Z_f(s)$ if there exists an arbitrarily small neighbourhood of 0 for which it is a pole.

The following lemma is trivial.

(4.1.3) Lemma. *Suppose that we have blown up k times but we have not yet an embedded resolution. Let P be a point at which $f \circ g_1 \circ \cdots \circ g_k$ has not normal crossings. Let μ be the multiplicity in P of the strict transform of f around P and let g_{k+1} be the blowing-up at P.*

(a) Suppose that two exceptional curves E_i and E_j contain P. Then $-\nu_{k+1}/N_{k+1}$ is equal to $-(\nu_i+\nu_j)/(N_i+N_j+\mu)$ and this is larger than $\min\{-\nu_i/N_i, -\nu_j/N_j\}$.

(b) Suppose that exactly one exceptional curve E_i contains P and that $\mu \geq 2$. Then E_{k+1} has numerical data $(N_i+\mu, \nu_i+1)$ and $-(\nu_i+1)/(N_i+\mu)$ is in between

$-1/\mu$ *and* $-\nu_i/N_i$.

(c) Suppose that exactly one exceptional curve E_i *contains* P *and that* $\mu = 1$. *Remark that the two curves are tangent at* P *because we do not have normal crossings at* P. *Let* g_{k+2} *be the blowing-up at* $E_i \cap E_{k+1}$. *Remark that we do not have to blow up at a point of* E_{k+1} *anymore. The numerical data of* E_{k+2} *are* $(2N_i + 2, 2\nu_i + 1)$, *and* $-(2\nu_i + 1)/(2N_i + 2)$ *is in between* $-1/2$ *and* $-\nu_i/N_i$. *Let* s_0 *be a candidate pole associated to* E_{k+1}. *Because* s_0 *is not a candidate pole associated to* E_{k+2}, *which is a consequence of* $-\nu_{k+1}/N_{k+1} \neq -\nu_{k+2}/N_{k+2}$, *the contribution of* E_{k+1} *to the coefficient* b_{-2} *in the MacLaurin series of* $Z_f(s)$ *at* s_0 *is zero. It follows from the proposition in* 3.1 *that* E_{k+1} *does not give a contribution to the residue* b_{-1} *of* $Z_f(s)$ *at* s_0.

(4.1.4) Suppose that after some blowing-ups, the pullback of f has not normal crossings at a point P. Suppose also that the real parts of the candidate poles associated to the exceptional curves through P are all larger than or equal to $-1/2$. Then it follows from the above lemma that the components above P in the final resolution do not give a contribution to a candidate pole with real part less than $-1/2$.

Corollary. *Zeta functions of convergent power series of multiplicity at least* 4 *do not have a pole with real part in* $]-\infty, -1/2[\setminus\{-1\}$.

Indeed, every exceptional curve in the minimal embedded resolution of f lies above a point of E_1 (considered in the stage when it is created), which has a candidate pole with real part larger than or equal to $-1/2$.

(4.1.5) To deal with multiplicity 2 and 3, we will study an 'easier' element of $K{\ll}x, y{\gg}$. We will use the following theorem (see [Ig3, Theorem 2.3.1]).

Weierstrass Preparation Theorem.
If $f(z_1, \ldots, z_{n-1}, w) = f(z, w) \in K{\ll}z, w{\gg}$ is not identically zero on the w-axis, then f can be written uniquely as $f = (w^e + a_1(z)w^{e-1} + \cdots + a_e(z))h$, where $a_i(z) \in K{\ll}z{\gg}$ satisfies $a_i(0) = 0$ and $h \in K{\ll}z, w{\gg}$ satisfies $h(0) \neq 0$.

Because $h(0) \neq 0$ implies that $|h|$ is constant on a neighbourhood of 0, we have that Igusa's p-adic zeta functions of f and $w^e + a_1(z)w^{e-1} + \cdots + a_e(z)$ have the same poles. After an appropriate coordinate transformation, the desired form will appear. For example, the coordinate transformation $(z, w) \mapsto (z, w - a_1(z)/e)$ cancels the term $a_1(z)w^{e-1}$.

(4.1.6) *Example.* Let $f \in K{\ll}x, y{\gg}$ have multiplicity 3 and let $f_3 = y^3 + xy^2 = y^2(y + x)$ be the homogeneous part of f of degree 3. By the Weierstrass preparation theorem, we may work with a function of the form $y^3 + a_1(x)y^2 + a_2(x)y + a_3(x)$, with $\mathrm{mult}(a_1(x)) = 1$, $\mathrm{mult}(a_2(x)) \geq 3$ and $\mathrm{mult}(a_3(x)) \geq 4$. One can check that there exists a coordinate transformation $(x, y) \mapsto (x, y - k(x))$ such that the function becomes of the form $y^3 + b_1(x)y^2 + b_3(x)$, with $\mathrm{mult}(b_1(x)) = 1$ and $\mathrm{mult}(b_3(x)) \geq 4$. After another coordinate transformation, we get the form $y^3 + xy^2 + g(x)$, with $\mathrm{mult}(g(x)) \geq 4$.

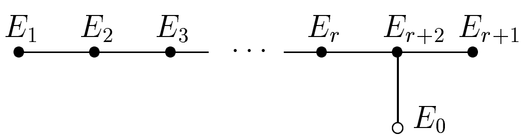
(4.1.7) Theorem. *We have*

$$\mathcal{P}_2^K \cap \left]-\infty, -\frac{1}{2}\right[= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}$$

and every Igusa's p-adic zeta function has at most one pole in $]-1, -1/2]$. *Moreover, if* $f \in K\ll x, y\gg$ *has multiplicity at least* 4, *then* $Z_f(s)$ *has no pole with real part in* $]-\infty, -1/2[\setminus\{-1\}$.

Proof. Because the calculations are analogous to the calculations in [SV] for the local topological zeta function, we do not treat all the cases in this paper.

(a) Suppose that f is an element of $K \ll x_1, x_2 \gg$ with multiplicity 2. When we apply the ideas of (4.1.5), we see that it is enough to consider x_1^2 and $x_1^2 + ax_2^l$, with $l \in \mathbb{Z}_{>1}$ and $a \in K^\times$. If $f = x_1^2$, the candidate poles of $Z_f(s)$ are $-1/2 + (k\pi\sqrt{-1})/(\log p)$, $k \in \mathbb{Z}$. If $l = 2$, the calculations are analogous as in (2.9). If l is odd, write $l = 2r + 1$. After r blowing-ups, the strict transform of $f^{-1}\{0\}$ is nonsingular and tangent to E_r. The numerical data of E_i, $i = 1, \ldots, r$, are $(2i, i + 1)$. To get the minimal embedded resolution, we now blow up twice. Let E_0 be the strict transform of $f^{-1}\{0\}$. Remark that $T = \{0, 1, \ldots, r\}$. The dual resolution graph and the numerical data are given below.



$E_1(2,2)$ $\quad$ $E_r(2r, r+1)$
$E_2(4,3)$ $\quad$ $E_{r+1}(2r+1, r+2)$
$E_3(6,4)$ $\quad$ $E_{r+2}(4r+2, 2r+3)$

It follows from section 3.1 that the candidate poles associated to $E_1, \ldots, E_{r+1}$ are not poles. The other candidate poles have real part -1 or $-(2r+3)/(4r+2) = -1/2 - 1/(2r+1)$. We calculate the residue of $Z_f(s)$ at the candidate pole $s_0 = -1/2 - 1/(2r+1)$. Because

$$\begin{aligned}
&\left[\int_{aR} |y_1|^{(2r+1)s+r+1}|y_1 + a|^s|dy_1|\right]^{\mathrm{mc}}_{s=s_0} \\
&= |a|^{-1/(2r+1)} \left[\int_R |y|^{(2r+1)s+r+1}|y+1|^s|dy|\right]^{\mathrm{mc}}_{s=s_0} \\
&= |a|^{-1/(2r+1)} \left[\int_{R\setminus(-1+P)} |y|^{(2r+1)s+r+1}|dy| + \int_{-1+P} |y+1|^s|dy|\right]^{\mathrm{mc}}_{s=s_0} \\
&= |a|^{-1/(2r+1)} \left(\frac{q-2}{q} + \frac{q-1}{q}\frac{1}{q^{\alpha_{r+1}}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_0}-1}\right)
\end{aligned}$$

and

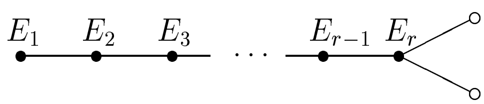
$$
\begin{aligned}
\left[\int_{\frac{1}{a}P} |y_2|^{2rs+r}|1+ay_2|^s|dy_2|\right]^{\mathrm{mc}}_{s=s_0} &= |a|^{-1/(2r+1)}\left[\int_P |y|^{2rs+r}|1+y|^s|dy|\right]^{\mathrm{mc}}_{s=s_0} \\
&= |a|^{-1/(2r+1)}\left[\int_P |y|^{2rs+r}|dy|\right]^{\mathrm{mc}}_{s=s_0} \\
&= |a|^{-1/(2r+1)}\frac{q-1}{q}\frac{1}{q^{\alpha_r}-1}
\end{aligned}
$$

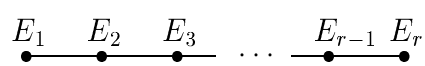
the residue of $Z_f(s)$ at the candidate pole $s_0 = -1/2 - 1/(2r+1)$ is

$$
|a|^{-1/(2r+1)}\left(\frac{q-2}{q} + \frac{q-1}{q}\frac{1}{q^{\alpha_{r+1}}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_0}-1} + \frac{q-1}{q}\frac{1}{q^{\alpha_r}-1}\right)
$$

multiplied by a factor different from zero (see Remark (ii) in (2.7)). Because $\alpha_{r+1} = (2r+1)s_0 + r + 2 = 1/2 > 0$, $\alpha_0 = s_0 + 1 = 1/2 - 1/(2r+1) > 0$ and $\alpha_r = 2rs_0 + r + 1 = 1/(2r+1) > 0$, we have that the last three terms of this expression are strictly positive. Consequently the whole expression is strictly positive and thus different from zero, so that $-1/2 - 1/(2r+1)$ is a pole of $Z_f(s)$.

If l is even and larger than 2, write $l = 2r$. We have to blow up r times to obtain an embedded resolution. We have $E_1(2,2)$, $E_2(4,3)$, $E_3(6,4)$, $\ldots$, $E_{r-1}(2r-2, r)$, $E_r(2r, r+1)$. We obtain the first dual resolution graph if $-a$ is a square in K. Otherwise, we obtain the second dual resolution graph.





It follows from section 3.1 that the candidate poles associated to $E_1, \ldots, E_{r-1}$ are not poles. The other candidate poles have real part -1 or $-(r+1)/(2r) = -1/2 - 1/(2r)$ in the first case and $-(r+1)/(2r) = -1/2 - 1/(2r)$ in the second case. Now we prove that $-1/2 - 1/(2r)$ is an element of $\mathcal{P}_2^K$. Suppose first that $p \neq 2$. Then there exists an element a of K with norm 1 for which $-a$ is not a square in K. For such an a, the residue of $Z_f(s)$ at $s_0 = -1/2 - 1/(2r)$ is the nonzero factor times

$$
\frac{q-1}{q}\frac{1}{q^{\alpha_{r-1}}-1} + 1.
$$

Suppose now that $p = 2$. Remark that every element of the residue field is a square in this case. Let $b \in R^\times$. If $b' \in b + P$, then $b'^2 - b^2 \in P^2$. Consequently, there exists an $a \in -b^2 + P$ such that $|a + x^2| = 1/q$ for all $x \in b + P$. For such an a, the residue of $Z_f(s)$ at $s_0 = -1/2 - 1/(2r)$ is the nonzero factor times

$$
\frac{q-1}{q}\frac{1}{q^{\alpha_{r-1}}-1} + \frac{q-1}{q} + \frac{1}{q}\left(\frac{1}{q}\right)^{s_0}.
$$

Because $\alpha_{r-1} = (2r-2)s_0 + r = 1/r > 0$, we obtain in the two cases that this residue is strictly positive, which implies that $-1/2 - 1/(2r)$ is a pole.

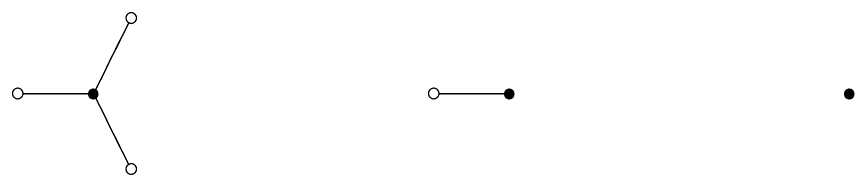
Our conclusion of part (a) is thus

$$\begin{aligned} \{s_0 \mid \exists f \in K\langle\langle x_1, x_2\rangle\rangle \quad &: \quad \operatorname{mult}(f) = 2 \text{ and } Z_f(s) \text{ has a pole with real part } s_0\} \\ &= \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\} \cup \left\{ -\frac{1}{2} \right\}. \end{aligned}$$

Remark that Newton polyhedra could also be used to deal with (a), see [DH].

(b) Suppose that f is an element of $K\langle\langle x_1, x_2\rangle\rangle$ with multiplicity 3. Up to an affine coordinate transformation, there are three cases for f_3.

We consider the case that f_3 is a product of three different linear factors over $K^{\text{alg cl}}$. Then we obtain an embedded resolution after one blowing-up. There are three possibilities for the dual resolution graph, depending on whether f_3 splits into linear factors over K, f_3 is a product of a linear factor and an irreducible factor of degree 2 over K or f_3 is irreducible over K. The dual resolution graphs are respectively



The equations of $f_3 \circ g$ in the charts determined by $(y_1, y_2) \mapsto (y_1, y_1 y_2)$ and $(z_1, z_2) \mapsto (z_1 z_2, z_2)$ are respectively of the form $y_1^3 h_1$ and $z_2^3 h_2$. In the last case for example, we have that h_1 and h_2 are nonvanishing on the exceptional curve. The real parts of the candidate poles of $Z_f(s)$ are -1 and $-2/3 = -1/2 - 1/6$ in the first two cases and $-2/3 = -1/2 - 1/6$ in the last case.

The other cases are treated in [SV] for the topological zeta function and are very similar for Igusa's p-adic zeta function.

(c) Suppose that f is an element of $K\langle\langle x_1, x_2\rangle\rangle$ with multiplicity at least 4. We explained in (4.1.4) that $Z_f(s)$ has no pole with real part in $]-\infty, -1/2[\setminus\{-1\}$. $\square$

(4.1.8) Let χ be a character of $R^\times$. For $n \in \mathbb{Z}_{>0}$, we define the set $\mathcal{P}^K_{n,\chi}$ by

$$\mathcal{P}^K_{n,\chi} := \{s_0 \mid \exists f \in F^K_n \;:\; Z_{f,\chi}(s) \textit{ has a pole with real part } s_0\}.$$

Theorem. *We have*

$$\mathcal{P}^K_{2,\chi} \cap \left]-\infty, -\frac{1}{2}\right[\subset \left\{ -\frac{1}{2} - \frac{1}{i} \,\middle|\, i \in \mathbb{Z}_{>1} \right\}$$

and every Igusa's p-adic zeta function has at most one pole in $]-1, -1/2]$.

Proof. In the proof of the previous theorem we needed only the proposition in 3.1 to obtain that some candidate poles were not poles. Now we work with a character and we can use the same proposition to prove that these candidate poles are not poles. $\square$

4.2 Surfaces

In this section, we prove the following theorem.

(4.2.0) Theorem. *We have*

$$\mathcal{P}_3^K \cap]-\infty,-1[=\left\{-1-\frac{1}{i}\,\middle|\, i\in\mathbb{Z}_{>1}\right\}.$$

Moreover, if $f\in K\ll x,y,z\gg$ has multiplicity at least 3, then $Z_f(s)$ has no pole with real part less than -1.

Remark. (i) It is a priori not obvious that the smallest value of $\mathcal{P}_3^K$ is $-3/2$. This is in contrast with the fact that it easily follows from lemma 4.1.3 that the smallest value of $\mathcal{P}_2^K$ is -1.
(ii) Let χ be a character of $R^\times$. Then one proves analogously as below that an element of $\mathcal{P}_{3,\chi}^K$ less than -1 is of the form $-1-1/i$, $i\in\mathbb{Z}_{>1}$. Using the remark in section 3.2, the arguments below will also imply that $Z_{f,\chi}(s)$ has no pole with real part less than -1 if $f\in K\ll x,y,z\gg$ has multiplicity at least 3.

4.2.1 Multiplicity 2

(4.2.1.1) Let $f(x)$, $x=(x_1,\ldots,x_n)$, be a K-analytic function on an open and compact subset X of K^n. Let $g(y)$, $y=(y_1,\ldots,y_m)$, be a K-analytic function on an open and compact subset Y of K^m. Then $f(x)+g(y)$ is a K-analytic function on the open and compact subset $X\times Y$ of K^{n+m}. Put $A(s,\rho):=q^{s+1}-1$ if ρ is the trivial character of $R^\times$ and $A(s,\rho):=1$ if ρ is another character of $R^\times$.

Fix a character χ of $R^\times$. Suppose that the only critical value of f and g is zero. Then the poles of $A(s,\chi)Z_{f+g,\chi}(s)$ are of the form s_1+s_2 with s_1 a pole of $A(s,\chi')Z_{f,\chi'}(s)$ and s_2 a pole of $A(s,\chi'')Z_{g,\chi''}(s)$ for some characters χ' and χ'' of $R^\times$ satisfying $\chi'\chi''=\chi$ (see [Ig1] or [De2, (5.1)]).

(4.2.1.2) Proposition. *The set*

$$\{s_0\mid \exists f\in K\ll x,y,z\gg:\ mult(f)=2 \text{ and } Z_f(s) \text{ has a pole in } s_0\}\cap]-\infty,-1[$$

is equal to

$$\left\{-1-\frac{1}{i}\,\middle|\, i\in\mathbb{Z}_{>1}\right\}.$$

Proof. Let f be an element of $K{\ll}x,y,z{\gg}$ with multiplicity 2. Up to an affine coordinate transformation, the part of degree 2 of f is equal to $ax^2+by^2+cz^2$, with $a,b,c\in K$ and $a\neq 0$. Using (4.1.5), we may suppose that f is of the form $x^2+g(y,z)$ with $g(y,z)\in K{\ll}y,z{\gg}$. The statement in (4.2.1.1) and the result for curves imply that every pole of $Z_f(s)$ less than -1 is of the form $-1-1/i$, $i\in\mathbb{Z}_{>1}$.

Now we prove the other inclusion. Using the p-adic stationary phase formula [Ig3, Theorem 10.2.1], we obtain that Igusa's p-adic zeta function of $xy+z^i$, $i\geq 2$, is equal to

$$\left(\frac{q-1}{q}\right)\left(\frac{1-q^{-s-3}+(q-1)(q^{-2s-4}+q^{-3s-5}+\cdots+q^{-(i-1)s-(i+1)})}{(1-q^{-s-1})(1-q^{-is-(i+1)})}\right).$$

The real poles of this zeta function are -1 and $-1-1/i$. ☐

4.2.2 Multiplicity larger than 2

(4.2.2.1) Let f be an element of $K{\ll}x,y,z{\gg}$. Fix a (small enough) neighbourhood X of $0\in K^3$ on which f is convergent and an embedded resolution $g:Y\to X$ of f which is a K-bianalytic map at the points where f has normal crossings and which is a composition of blowing-ups $g_{ij}:X_i\to X_j$ with centre a K-analytic closed submanifold D_j and with exceptional surface E_i satisfying:

(a) the codimension of D_j in X_j is at least 2;

(b) D_j is a subset of the zero locus of the strict transform of f on each chart (the strict transform of f is not defined globally);

(c) the union of the exceptional varieties in X_j has only normal crossings with D_j, i.e., for all $P\in D_j$, there are three surface germs through P which are in normal crossings such that each exceptional surface germ through P is one of them and such that the germ of D_j at P is the intersection of some of them;

(d) the image of D_j in $X\subset K^3$ contains the origin of K^3; and

(e) D_j contains a point in which the pullback of f has not normal crossings.

Remark that such a resolution always exists by Hironaka's theorem [Hi].

(4.2.2.2) The following table gives the numerical data of E_i. In the columns, the dimension of D_j is kept fixed. In the rows, the number of exceptional surfaces through D_j is kept fixed. So E_k, E_l and E_m represent exceptional surfaces that contain D_j. The multiplicity of the strict transform of f in D_j is denoted by μ_{D_j}.

	D_j is a point P	D_j is a curve L
/	$(\mu_P,3)$	$(\mu_L,2)$
E_k	$(N_k+\mu_P,\nu_k+2)$	$(N_k+\mu_L,\nu_k+1)$
E_k and E_l	$(N_k+N_l+\mu_P,\nu_k+\nu_l+1)$	$(N_k+N_l+\mu_L,\nu_k+\nu_l)$
E_k, E_l and E_m	$(N_k+N_l+N_m+\mu_P,\nu_k+\nu_l+\nu_m)$	/

(4.2.2.3) Lemma. *Suppose that* $\mathrm{mult}(f) \geq 3$. *If there is no exceptional surface through* D_j, *then* $-\nu_i/N_i \geq -1$.

Proof. The analogous statement for the local topological zeta function is treated in [SV, (3.3.3)]. The proof of the lemma is a trivial adaptation of the proof there. □

(4.2.2.4) Suppose that D_j is contained in at least one exceptional surface and that the real parts of the candidate poles associated to the exceptional surfaces that pass through D_j are larger than or equal to -1. Then the table in (4.2.2.2) implies that also $-\nu_i/N_i \geq -1$, unless D_j is a regular point P of the strict transform of f around P through which only one exceptional surface E_0 passes and $-\nu_0/N_0 = -1$. Suppose that we are in this situation. Let Z_0 be a (small enough) neighbourhood of P such that, if we restrict the blowing-ups g_{ij} to the inverse image of Z_0, we get an embedded resolution $h = h_1 \circ \cdots \circ h_s$ of the pullback of f which is a composition of blowing-ups $h_i : Z_i \to Z_{i-1}$, $i \in \{1, \ldots, s\}$, with centre $D'_{i-1} := D_{i-1} \cap Z_{i-1}$ and exceptional surface $E'_i := E_i \cap Z_i$ for which P is in the image of D'_{i-1} under $h_1 \circ \cdots \circ h_{i-1}$.

Remark that it can happen that g_{ij} is a K-bianalytic map on the inverse image of Z_0. Because we did not specify the indices in (4.2.2.1), we were able to get a nice notation here. From now on, we study the resolution $h : Z_s \to Z_0$ of the pullback of f.

Lemma. *(a) If* $D_i = D'_i$, *then* D_i *is a subset of* $E'_0 := E_0 \cap Z_0$.

(b) Suppose that $\mathrm{mult}(f) \geq 3$. *Then we have* $\nu_i \leq N_i + 1$ *for every exceptional surface* E_i, $i \in \{1, \ldots, s\}$. *Moreover,* $\nu_i = N_i + 1$ *if and only if* D_{i-1} *is a point and the numerical data of every exceptional surface* E_j *different from* E_0 *and through* D_{i-1} *satisfy* $\nu_j = N_j + 1$.

(c) If $\mathrm{mult}(f) \geq 3$ *and if the numerical data of* E_i *satisfy* $\nu_i = N_i + 1$, *then* $-\nu_i/N_i \neq -\nu_j/N_j$ *for every exceptional surface* E_j *that intersects* E_i *at some stage of the resolution process.*

Proof. See [SV, (3.3.5),(3.3.6) and (3.3.7)]. □

Proposition. *If* $\mathrm{mult}(f) \geq 3$, *then* $Z_f(s)$ *has no pole with real part less than* -1.

Proof. The proof is analogous to the one in [SV, (3.3.8)]. Now we have to use the proposition in 3.2. □

Appendix. Poles and divisibility of the M_i

Suppose that f is a K-analytic function on R^n defined by a power series over

R which is convergent on the whole of R^n. Let l be the smallest real part of a pole of $Z_f(s)$ and let M_i be the number of solutions of $f(x) \equiv 0 \bmod P^i$ in $(R/P^i)^n$.

Proposition. *There exists an integer a which is independent of i such that M_i is an integer multiple of $q^{\ulcorner (n+l)i-a \urcorner}$ for all $i \in \mathbb{Z}_{\geq 0}$.*

Remark. (i) The number $\ulcorner (n+l)i-a \urcorner$ is the smallest integer larger than or equal to $(n+l)i-a$, which rises $(n+l>0)$ linearly as a function of i with a slope depending on l.
(ii) The statement is trivial if $(n+l)i-a \leq 0$ because the M_i are integers. If $(n+l)i-a>0$, which is the case for i large enough, it claims that M_i is divisible by $q^{\ulcorner (n+l)i-a \urcorner}$.

Proof. Put $t=q^{-s}$. It follows from (2.2) that we can write

$$Z_f(t) = \frac{A(t)}{\prod_{j\in T}\left(1-q^{-\nu_j}t^{N_j}\right)},$$

where $A(t)$ is a polynomial with coefficients in the set $S := \{z/q^i \mid z \in \mathbb{Z}, i \in \mathbb{Z}_{\geq 0}\}$. By using the division algorithm for polynomials we can write

$$Z_f(t) = \frac{B(t)}{\prod_{j\in K}\left(1-q^{-\nu_j}t^{N_j}\right)},$$

where $B(t)$ is a polynomial with coefficients in S and where $K := \{j \in T \mid -\nu_j/N_j \geq l\}$.

The Poincaré series $P(t)$ of f is defined by

$$P(t) = \sum_{i=0}^{\infty} N_i \frac{t^i}{q^{ni}}.$$

and can be obtained from $Z_f(t)$ by the relation

$$P(t) = \frac{1-tZ_f(t)}{1-t}.$$

It easily follows from the defining integral of Igusa's p-adic zeta function that $Z_f(t=1)=1$. Consequently, $1-tZ_f(t)$ is divisible by $1-t$ and $P(t)$ can be written as

$$P(t) = \frac{C(t)}{\prod_{j\in K}\left(1-q^{-\nu_j}t^{N_j}\right)},$$

where $C(t)$ is a polynomial with coefficients in S.

We will say that a formal power series in t has the divisibility property if the coefficient of t^i/q^{ni} is an integer multiple of $q^{\ulcorner (n+l)i \urcorner}$ for every i.

For $j \in K$, the series

$$\frac{1}{1-q^{-\nu_j}t^{N_j}} = \sum_{i=0}^{\infty} q^{-i\nu_j}t^{iN_j} = \sum_{i=0}^{\infty} q^{i(nN_j-\nu_j)}\frac{t^{iN_j}}{q^{niN_j}}$$

has the divisibility property because $nN_j - \nu_j$ is an integer larger than or equal to $N_j(n+l)$. Let a be an integer such that the polynomial $D(t) := q^a C(t)$ has the divisibility property. Remark that $C(t) = q^{-a} D(t)$.

One can easily check that the product of a finite number of power series with the divisibility property also has the divisibility property. This implies that $P(t)$ is a power series with the divisibility property, multiplied by q^{-a}. Hence M_i is an integer multiple of $q^{\ulcorner (n+l)i \urcorner - a} = q^{\ulcorner (n+l)i - a \urcorner}$ for all i. $\square$

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