A vanishing result for Igusa's p-adic zeta functions with character

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Abstract

Let K be a p-adic field and let f be a K-analytic function on an open and compact subset of K^3. Let R be the valuation ring of K and let χ be an arbitrary character of $R^\times$. Let $Z_{f,\chi}(s)$ be Igusa's p-adic zeta function. In this paper, we prove a vanishing result for candidate poles of $Z_{f,\chi}(s)$. This result implies that $Z_{f,\chi}(s)$ has no pole with real part less than -1 if f has no point of multiplicity 2.

1 Introduction

(1.1) Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed uniformizing parameter for R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z, $|z| = q^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z = z\pi^{-\operatorname{ord} z}$ the angular component of z.

Let χ be a character of $R^\times$, i.e., a homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. We formally put $\chi(0) = 0$. Let e be the conductor of χ, i.e., the smallest $a \in \mathbb{Z}_{>0}$ such that χ is trivial on $1 + P^a$.

(1.2) Let f be a K-analytic function on an open and compact subset X of K^n and put $x = (x_1, \ldots, x_n)$. Igusa's p-adic zeta function of f and χ is defined by

$$Z_{f,\chi}(s) = \int_X \chi(\operatorname{ac} f(x))|f(x)|^s\,|dx|$$

for $s \in \mathbb{C}$, $\operatorname{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on K^n, so normalized that R^n has measure 1. Igusa proved that it is a rational function of q^{-s}, so that it extends to a meromorphic function $Z_{f,\chi}(s)$ on $\mathbb{C}$ which is also called Igusa's p-adic zeta function of f. If χ is the trivial character, we will also write $Z_f(s)$.

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(1.3) Let $g : Y = Y_t \to X = Y_0$ be an embedded resolution of f which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$. Suppose that each g_i is a blowing-up along a K-analytic closed submanifold of codimension larger than one which has only normal crossings with the union of the exceptional varieties of $g_1 \circ \cdots \circ g_{i-1}$. The exceptional variety of g_i and also the strict transforms of this variety are denoted by E_i. The multiplicities of $f \circ g$ and g^*dx along E_i are respectively denoted by N_i and $\nu_i - 1$. Note that such a resolution always exists by Hironaka's theorem [Hi].

If one has an embedded resolution of f, one can write down a set of candidate poles of $Z_{f,\chi}(s)$ which contains all poles of $Z_{f,\chi}(s)$. Candidate poles are associated to a component of the strict transform of $f^{-1}\{0\}$ or to an exceptional variety E_i. One associates candidate poles to an exceptional variety E_i if $\chi^{N_i} = 1$, and in this case, these candidate poles are $-\nu_i/N_i + (2k\pi\sqrt{-1})/(N_i \log q)$, with $k \in \mathbb{Z}$. Most candidate poles are actually not poles. This would be elucidated if the monodromy conjecture [De] is true, see for example [Lo], [Ve] and [ACLM]. In order to prove that a candidate pole s_0 of expected order 1 is not a pole, we have to prove that the residue of $Z_{f,\chi}(s)$ at s_0 is zero. We recall a formula for this residue which we will use in this paper.

Let s_0 be a candidate pole of $Z_{f,\chi}(s)$ of expected order 1. Let E_r, $r \in \{1, \ldots, t\}$, be an exceptional variety with candidate pole s_0 (and thus also with $\chi^{N_r} = 1$). Let (V, z) be a compact chart on Y_r such that $z_n = 0$ is an equation of E_r on V. Write

$$f \circ g_1 \circ \cdots \circ g_r = \alpha z_n^{N_r} \quad \text{and} \quad (g_1 \circ \cdots \circ g_r)^* dx = \beta z_n^{\nu_r - 1} dz$$

on V, for K-analytic functions α and β on V. We have that $\overline{z} = (z_1, \ldots, z_{n-1})$ determines coordinates on the closed submanifold $\overline{V} = V \cap E_r$ which is defined by $z_n = 0$. Consider the volume form $d\overline{z} = dz_1 \wedge \cdots \wedge dz_{n-1}$ on $\overline{V}$. We proved in [Se1, (2.6)] that the contribution of the strict transform of $\overline{V}$ in Y to the residue of $Z_{f,\chi}(s)$ at s_0 is equal to

$$\left(\frac{q-1}{qN_r \log q} \right) \left[\int_{\overline{V}} \chi(\mathrm{ac}\,\alpha) |\alpha|^s |\beta| |d\overline{z}| \right]^{\mathrm{mc}}_{s=s_0} .$$

Here, $[\cdot]^{\mathrm{mc}}_{s=s_0}$ is the meromorphic continuation of the function between the brackets evaluated at $s = s_0$.

(1.4) Let f be a K-analytic function on an open and compact subset X of K^3 and let χ be an arbitrary character of $R^\times$. We proved in [Se1] that the real part of a pole of $Z_{f,\chi}(s)$ is of the form $-1 - 1/i$, with $i \in \mathbb{Z}_{>1}$, if it is less than -1. Moreover, we proved that $Z_f(s)$ has no pole with real part less than -1 if f has no point of multiplicity 2. This result is also valid for $Z_{f,\chi}(s)$:

Theorem. We have that $Z_{f,\chi}(s)$ has no pole with real part less than -1 if f has no point of multiplicity 2.

This follows immediately from [Se1, (4.2.2.4)], from the proof in [SV, (3.3.8)] adapted to this context and from the following vanishing result.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2, y_3))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part of the form $ey_1^k y_2^l y_3^m (y_1 + y_2)^n$, with $e \in K^\times$ and $k, l, m, n \in \mathbb{Z}_{\geq 0}$. Then the contribution of E_r to the residue of $Z_{f,\chi}(s)$ at s_0 is zero.*

We have already proved this proposition for the trivial character in [Se1, Section 3.2]. The other cases will be treated in Section 3 of this paper by using the formula for the residue in (1.3). Most vanishing results with character are up till now only for characters with conductor 1. In this paper, we treat characters with arbitrary conductor. If we would have restricted us to characters with conductor 1, we would need well known results on character sums of characters $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$. To treat characters with arbitrary conductor, we need results on character sums of characters $\chi : (R/P^e)^\times \to \mathbb{C}^\times$. This is treated in Section 2 in a slightly more general context.

2 Character sums

(2.1) Let $G, .$ be a finite group. Let $\mathbb{C}^\times, .$ be the multiplicative group of the field of complex numbers. A character χ of G is a group homomorphism $\chi : G \to \mathbb{C}^\times$. Note that $\chi(x)$ is a $(|G|)$th root of unity for every $x \in G$.

Let R be a discrete valuation ring, abbreviated DVR. Let P be the maximal ideal of R and suppose that the residue field R/P is isomorphic to $\mathbb{F}_q$. Let π be a fixed uniformizing parameter for R. A character χ of the group $R^\times, .$ is a group homomorphism $\chi : R^\times \to \mathbb{C}^\times$ with finite image. The conductor $e_\chi = e$ of χ is the smallest $u \in \mathbb{Z}_{>0}$ such that χ is trivial on $1 + P^u$.

Valuation rings of p-adic fields are the DVRs which are interesting for our purposes. Other interesting DVRs with finite residue field are the rings of formal power series $\mathbb{F}_q[[t]]$.

For every $u \in \mathbb{Z}_{>0}$, there is a natural one to one correspondence between characters $\chi : (R/P^u)^\times \to \mathbb{C}^\times$ and characters $\chi : R^\times \to \mathbb{C}^\times$ with conductor less than or equal to u.

Let $u \in \mathbb{Z}_{>0}$ and let $L \subset R$ be a union of cosets of P^u. By abuse of notation, we will consider L sometimes as a subset of R/P^u. We will write $L \subset R/P^u$ if we want to stress this. If all elements of $L \subset R/P^u$ are units, we will also write $L \subset (R/P^u)^\times$.

The characters of a finite group $G, .$ (and of $R^\times, .$) can be multiplied in an obvious way. The set of characters becomes a group for this operation. The identity of this group is the constant map on 1, and this character is called the trivial character.

We now give a lot of propositions on character sums which will be used in Section 3. In [IR, Chapter 8], character sums of $\mathbb{F}_p^\times$ are treated. We will use similar techniques in our proofs.

(2.2) Proposition. *Let χ be a non-trivial character of a finite group G. Then $\sum_{x\in G}\chi(x) = 0$.*

Proof. Fix $a \in G$ such that $\chi(a) \neq 1$. Then

$$\chi(a)\sum_{x\in G}\chi(x) = \sum_{x\in G}\chi(ax) = \sum_{x\in G}\chi(x).$$

The last equality is a consequence of the fact that ax runs over all element of G if x does. Our statement follows because $\chi(a) \neq 1$. □

The previous proposition is well known. Now comes the serious work.

(2.3) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor e. Then*

$$\begin{aligned} \sum_{x\in(R/P^e)^\times} \chi(x) &= 0, \\ \sum_{\{(x_1,x_2)|x_1,x_2,x_1+x_2\in(R/P^e)^\times\}} \chi(x_1+x_2) &= 0. \end{aligned}$$

Proof. (1) This is Proposition 2.2 for $G = (R/P^e)^\times$.

(2) Every element of $(R/P^e)^\times$ can be written as x_1+x_2, with $x_1, x_2 \in (R/P^e)^\times$, in exactly $(q-2)q^{e-1}$ ways. Consequently,

$$\begin{aligned} \sum_{\{(x_1,x_2)|x_1,x_2,x_1+x_2\in(R/P^e)^\times\}} \chi(x_1+x_2) &= (q-2)q^{e-1}\sum_{t\in(R/P^e)^\times}\chi(t) \\ &= 0. \end{aligned}$$ □

(2.4) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor $e \geq 2$. Let $a \in R^\times$, let $i \in \{1, \dots, e-1\}$ and let $j \geq i$. Then*

$$\begin{aligned} \sum_{x\in a+P^i\subset(R/P^e)^\times} \chi(x) &= 0, \\ \sum_{x\in 1+P^i\subset(R/P^e)^\times} \chi(\pi^j a + x) &= 0, \\ \sum_{x\in(R/P^e)^\times} \chi(\pi^j a + x) &= 0. \end{aligned}$$

Proof. (1) Because

$$a + P^i \to 1 + P^i : x \mapsto a^{-1}x$$

is a bijection, we obtain

$$\begin{aligned}\sum_{x \in a+P^i \subset (R/P^e)^\times} \chi(x) &= \sum_{x \in a+P^i \subset (R/P^e)^\times} \chi(a)\chi(a^{-1}x) \\ &= \chi(a) \sum_{t \in 1+P^i \subset (R/P^e)^\times} \chi(t) \\ &= 0.\end{aligned}$$

The last equality follows from Proposition 2.2 because $1 + P^i$ is a subgroup of $(R/P^e)^\times$ on which χ is non-trivial.

(2) Because

$$1 + P^i \to 1 + P^i : x \mapsto \pi^j a + x$$

is a bijection, we obtain

$$\begin{aligned}\sum_{x \in 1+P^i \subset (R/P^e)^\times} \chi(\pi^j a + x) &= \sum_{t \in 1+P^i \subset (R/P^e)^\times} \chi(t) \\ &= 0.\end{aligned}$$

(3) The proof of the last equality is analogous to (2). □

(2.5) Proposition. *Let R be a DVR and let χ be a non-trivial character of $R^\times$ with conductor e. Let $a \in R^\times$ and let $i \in \{1, \ldots, e-1\}$. Then*

$$\begin{aligned}\sum_{x \in (R/P^e)^\times} \chi(x)\chi^{-1}(x + \pi^i a) &= \begin{cases} 0 & \text{if } i \in \{1, \ldots, e-2\} \\ -q^{e-1} & \text{if } i = e-1 \end{cases}, \\ \sum_{x \in (R/P^e)^\times \setminus (-a+P)} \chi(x)\chi^{-1}(x + a) &= \begin{cases} 0 & \text{if } e > 1 \\ -1 & \text{if } e = 1 \end{cases}, \\ \sum_{x \in (R/P^e)^\times} \chi(x)\chi^{-1}(\pi^i x + a) &= 0.\end{aligned}$$

Proof. (1) In this proof all calculations in R are modulo P^e. Because $\chi(x)\chi^{-1}(x+\pi^i a) = \chi(x/(x + \pi^i a))$, we study the values $x/(x + \pi^i a)$ if x runs over $(R/P^e)^\times$. We have that $x/(x + \pi^i a) = t$ if and only if $x(1 - t) = \pi^i at$, and such a t is of the form $t = 1 + \pi^i b$ for some $b \in R^\times$. Moreover the $x \in (R/P^e)^\times$ which satisfy this equation for such a fixed t are exactly the elements which are equal to $-ab^{-1}t$ modulo P^{e-i}. We thus have q^i values of $x \in (R/P^e)^\times$ for such a fixed t. Consequently

$$\sum_{x \in (R/P^e)^\times} \chi(x)\chi^{-1}(x + \pi^i a) = \sum_{x \in (R/P^e)^\times} \chi\left(\frac{x}{x + \pi^i a}\right)$$

$$
\begin{aligned}
&= q^i \sum_{t \in 1+(P^i \setminus P^{i+1}) \subset (R/P^e)^\times} \chi(t) \\
&= q^i \left(\sum_{t \in 1+P^i} \chi(t) - \sum_{t \in 1+P^{i+1}} \chi(t) \right) \\
&= \begin{cases} 0 & \text{if } i \in \{1, \ldots, e-2\} \\ -q^{e-1} & \text{if } i = e-1 \end{cases} .
\end{aligned}
$$

The last equality follows from Proposition 2.2 because $1+P^i$ and $1+P^{i+1}$ are subgroups of $(R/P^e)^\times$.

(2) One verifies easily that the map

$$
(R/P^e)^\times \setminus (-a+P) \to (R/P^e)^\times \setminus (1+P) : x \mapsto \frac{x}{x+a}
$$

is a bijection. Consequently

$$
\begin{aligned}
\sum_{x \in (R/P^e)^\times \setminus (-a+P)} \chi(x)\chi^{-1}(x+a) &= \sum_{x \in (R/P^e)^\times \setminus (-a+P)} \chi\left(\frac{x}{x+a}\right) \\
&= \sum_{t \in (R/P^e)^\times \setminus (1+P)} \chi(t) \\
&= \begin{cases} 0 & \text{if } e > 1 \\ -1 & \text{if } e = 1 \end{cases} .
\end{aligned}
$$

(3) One verifies easily that the map

$$
(R/P^e)^\times \to (R/P^e)^\times : x \mapsto \frac{x}{\pi^i x + a}
$$

is a bijection. Consequently

$$
\begin{aligned}
\sum_{x \in (R/P^e)^\times} \chi(x)\chi^{-1}(\pi^i x + a) &= \sum_{x \in (R/P^e)^\times} \chi\left(\frac{x}{\pi^i x + a}\right) \\
&= \sum_{t \in (R/P^e)^\times} \chi(t) \\
&= 0. \qquad \square
\end{aligned}
$$

From now on, e is always the maximum of the conductors of the characters which are involved.

(2.6) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi \neq \rho^{-1}$ and $\psi \neq \rho^{-1}$. Let $a \in R^\times$ and let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned} \sum_{x \in (R/P^e)^\times} \psi(x)\rho(x + \pi^i a) &= 0 \qquad \text{if} \quad e_\rho \leq e_{\psi\rho} + i - 1, \\ \sum_{x \in (R/P^e)^\times} \chi(x)\rho(\pi^i x + a) &= 0 \qquad \text{if} \quad e_\rho \leq e_\chi + i - 1. \end{aligned}$$

Proof. (1) Let $v \in 1 + P^{e_{\psi\rho}-1}$ such that $(\psi\rho)(v) \neq 1$. Then

$$\begin{aligned} \sum_{x \in (R/P^e)^\times} \psi(x)\rho(x + \pi^i a) &= \sum_{x \in (R/P^e)^\times} \psi(vx)\rho(vx + \pi^i a) \\ &= \sum_{x \in (R/P^e)^\times} \psi(vx)\rho(vx + \pi^i va) \\ &= (\psi\rho)(v) \sum_{x \in (R/P^e)^\times} \psi(x)\rho(x + \pi^i a). \end{aligned}$$

Because $(\psi\rho)(v) \neq 1$, we get our statement. In the first equality, we used a translation in the group $(R/P^e)^\times$. For the second equality, we used that $e_\rho \leq e_{\psi\rho} + i - 1$ and that $v \in 1 + P^{e_{\psi\rho}-1}$.

(2) Let $v \in 1 + P^{e_\chi - 1}$ such that $\chi(v) \neq 0$. Then

$$\begin{aligned} \sum_{x \in (R/P^e)^\times} \chi(x)\rho(\pi^i x + a) &= \sum_{x \in (R/P^e)^\times} \chi(vx)\rho(\pi^i vx + a) \\ &= \sum_{x \in (R/P^e)^\times} \chi(vx)\rho(\pi^i x + a) \\ &= \chi(v) \sum_{x \in (R/P^e)^\times} \chi(x)\rho(\pi^i x + a). \end{aligned}$$

Because $\chi(v) \neq 1$, we get our statement. In the first equality, we used a translation in the group $(R/P^e)^\times$. For the second equality, we used that $e_\rho \leq e_\chi + i - 1$ and that $v \in 1 + P^{e_\chi - 1}$. □

(2.7) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho = 1$. Note that the largest two values of e_χ, e_ψ, e_ρ are equal. Let $a, x_1, x_2 \in R^\times$ and let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned} \sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) &= \sum_{x \in (R/P^e)^\times} \chi(x)\rho(a + \pi^i x) \\ &= \sum_{x_1 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1), \end{aligned}$$

and this is equal to 0 if $e_\rho \leq e_\chi + i - 1$.

Proof. Fix $v \in (R/P^e)^\times$. Then

$$\begin{aligned}
&\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) \\
&\quad = (\chi\psi\rho)(v) \sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) \\
&\quad = \sum_{x_2 \in (R/P^e)^\times} \chi(vx_1)\psi(vx_2)\rho(avx_2 + \pi^i vx_1) \\
&\quad = \sum_{x_2 \in (R/P^e)^\times} \chi(vx_1)\psi(x_2)\rho(ax_2 + \pi^i vx_1).
\end{aligned}$$

Because vx_1 takes all values of $(R/P^e)^\times$ if v runs over $(R/P^e)^\times$, we obtain that

$$\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1)$$

is independent of $x_1 \in (R/P^e)^\times$. In the first equality, we put $x_1 = 1$:

$$\begin{aligned}
\sum_{x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) &= \sum_{x_2 \in (R/P^e)^\times} \psi(x_2)\rho(ax_2 + \pi^i) \\
&= \sum_{x \in (R/P^e)^\times} \psi(x^{-1})\rho(ax^{-1} + \pi^i) \\
&= \sum_{x \in (R/P^e)^\times} (\psi\rho)^{-1}(x)\rho(a + \pi^i x) \\
&= \sum_{x \in (R/P^e)^\times} \chi(x)\rho(a + \pi^i x).
\end{aligned}$$

This is the first equality we had to prove. Analogously as before, we obtain that

$$\sum_{x_1 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1)$$

is independent of x_2. If we put $x_2 = 1$, we obtain the second equality. We can use either of the equalities of (2.6) to prove that it is equal to 0 under the condition $e_\rho \leq e_\chi + i - 1$. □

(2.8) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho = 1$. Then*

$$\frac{1}{q^e} \sum_{x_1, x_2, x_1 + x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(x_1 + x_2) = \frac{q-1}{q} \sum_{x \in (R/P^e)^\times \setminus (-1+P)} \psi(x)\rho(1+x).$$

Proof. The map $\{(x_1,x_2) \mid x_1,x_2 \in (R/P^e)^\times \text{ and } x_2 \notin -1+P\} \to \{(x_1,x_2) \mid x_1,x_2,x_1+x_2 \in (R/P^e)^\times\} : (x_1,x_2) \mapsto (x_1,x_1x_2)$ is a bijection. Therefore

$$\begin{aligned}
&\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(x_1+x_2) \\
&\quad = \sum_{x_1,x_2\in(R/P^e)^\times;x_2\notin -1+P} \chi(x_1)\psi(x_1x_2)\rho(x_1+x_1x_2) \\
&\quad = \sum_{x_1,x_2\in(R/P^e)^\times;x_2\notin -1+P} \psi(x_2)\rho(1+x_2) \\
&\quad = (q-1)q^{e-1} \sum_{x\in(R/P^e)^\times\setminus(-1+P)} \psi(x)\rho(1+x). \qquad \square
\end{aligned}$$

(2.9) Proposition. *Let R be a DVR and let χ,ρ be non-trivial characters of $R^\times$ such that $\chi\rho \neq 1$. Let $a \in R^\times$. Let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2) &= 0, \\
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1) &= 0, \\
\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\rho(x_2) &= 0.
\end{aligned}$$

Proof. Let $v \in (R/P^e)^\times$ such that $(\chi\rho)(v) \neq 0$. The map $(x_1,x_2) \mapsto (vx_1,vx_2)$ is a bijection of $\{(x_1,x_2) \mid x_1,x_2 \in (R/P^e)^\times\}$ and also of $\{(x_1,x_2) \mid x_1,x_2,x_1+x_2 \in (R/P^e)^\times\}$. Therefore

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2) &= \sum_{x_1,x_2\in(R/P^e)^\times} \chi(vx_1)\rho(avx_1+\pi^i vx_2) \\
&= (\chi\rho)(v) \sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_1+\pi^i x_2),
\end{aligned}$$

$$\begin{aligned}
\sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1) &= \sum_{x_1,x_2\in(R/P^e)^\times} \chi(vx_1)\rho(avx_2+\pi^i vx_1) \\
&= (\chi\rho)(v) \sum_{x_1,x_2\in(R/P^e)^\times} \chi(x_1)\rho(ax_2+\pi^i x_1)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\rho(x_2) &= \sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(vx_1)\rho(vx_2) \\
&= (\chi\rho)(v) \sum_{x_1,x_2,x_1+x_2\in(R/P^e)^\times} \chi(x_1)\rho(x_2).
\end{aligned}$$

Because $(\chi\rho)(v) \neq 1$, we obtain our statements. $\square$

(2.10) Proposition. *Let R be a DVR and let χ, ψ, ρ be non-trivial characters of $R^\times$ such that $\chi\psi\rho \neq 1$. Let $a \in R^\times$. Let $i \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned} \sum_{x_1, x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(ax_2 + \pi^i x_1) &= 0, \\ \sum_{x_1, x_2, x_1+x_2 \in (R/P^e)^\times} \chi(x_1)\psi(x_2)\rho(x_1 + x_2) &= 0. \end{aligned}$$

Proof. We obtain these equalities analogously as in (2.9). Now we have to take $v \in (R/P^e)^\times$ such that $(\chi\psi\rho)(v) \neq 1$. $\square$

3 The vanishing result

Let K be a p-adic field, i.e., an extension of $\mathbb{Q}_p$ of finite degree. Let R be the valuation ring of K, P the maximal ideal of R, π a fixed uniformizing parameter for R and q the cardinality of the residue field R/P. For $z \in K$, let $\operatorname{ord} z \in \mathbb{Z} \cup \{+\infty\}$ denote the valuation of z, $|z| = q^{-\operatorname{ord} z}$ the absolute value of z and $\operatorname{ac} z = z\pi^{-\operatorname{ord} z}$ the angular component of z.

Let X be an open and compact subset of K^3. Let f be a K-analytic function on X. Let $g : Y = Y_t \to X = Y_0$ be an embedded resolution of f which is a composition $g_1 \circ \cdots \circ g_t$ of blowing-ups $g_i : Y_i \to Y_{i-1}$ with centre a K-analytic closed submanifold which has only normal crossings with the union of the exceptional surfaces in Y_{i-1} and with exceptional surface E_i. Let χ be a character of $R^\times$.

Proposition. *Let $r \in \{1, \ldots, t\}$ and let $P \in Y_{r-1}$ be the centre of the blowing-up g_r. Suppose that the expected order of a candidate pole s_0 associated to E_r is one. Suppose that there exists a chart $(V, y = (y_1, y_2, y_3))$ centred at P on which $f \circ g_1 \circ \cdots \circ g_{r-1}$ is given by a power series with lowest degree part of the form $ey_1^k y_2^l y_3^m (y_1 + y_2)^n$, with $e \in K^\times$ and $k, l, m, n \in \mathbb{Z}_{\geq 0}$. Then the contribution of $E_r \subset Y$ to the residue of $Z_{f,\chi}(s)$ at s_0 is zero.*

Proof. We may suppose that $f \circ g_1 \circ \cdots \circ g_{r-1} = ey_1^k y_2^l y_3^m (y_1 + y_2)^n + \theta$ and $(g_1 \circ \cdots \circ g_{r-1})^* dx = \rho y_1^{a-1} y_2^{b-1} y_3^{c-1} (y_1 + y_2)^{d-1} dy$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and ρ, θ K-analytic functions satisfying $\rho(0,0) \neq 0$ and $\operatorname{mult}(\theta) > k + l + m + n$. Remark that at least one of the numbers a, b, d is equal to 1.

We look at the chart $(O, z = (z_1, z_2, z_3))$ on Y_r for which $g_r(z_1, z_2, z_3) = (z_1 z_3, z_2 z_3, z_3)$. Then

$$f \circ g_1 \circ \cdots \circ g_r = z_3^{k+l+m+n} \left(e z_1^k z_2^l (z_1 + z_2)^n + z_3 \frac{\theta(z_1 z_3, z_2 z_3, z_3)}{z_3^{k+l+m+n+1}} \right)$$

and

$$(g_1 \circ \cdots \circ g_r)^* dx \;=\; \rho(z_1 z_3, z_2 z_3, z_3) z_1^{a-1} z_2^{b-1} z_3^{a+b+c+d-2} (z_1+z_2)^{d-1} dz.$$

Remark that the equation of E_r is $z_3 = 0$, that $N_r = k+l+m+n$ and that $\nu_r = a+b+c+d-1$. The contribution to the residue at s_0 of the strict transform in Y of an open and compact subset A of $E_r \subset Y_r$ which is contained in O is equal to $(q-1)/(qN_r \log q)$ times

$$\left[\int_A |e|^s |\rho(0,0,0)| \chi(\mathrm{ac}\, e)\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1 + z_2)|z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_1 \wedge dz_2|\right]^{\mathrm{mc}}_{s=s_0}.$$

Let $(O', z' = (z_1', z_2', z_3'))$ be the chart on Y_r for which $g_r(z_1', z_2', z_3') = (z_1' z_2', z_2', z_2' z_3')$. Analogously as before, we obtain that the contribution to the residue at s_0 of the strict transform in Y of an open and compact subset B of $E_r \subset Y_r$ which is contained in O' is equal to $(q-1)/(qN_r \log q)$ times

$$\left[\int_B |e|^s |\rho(0,0,0)| \chi(\mathrm{ac}\, e)\chi^k(\mathrm{ac}\, z_1)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, z_1 + 1)|z_1'|^{ks+a-1}|z_3'|^{ms+c-1}|z_1'+1|^{ns+d-1}|dz_1' \wedge dz_3'|\right]^{\mathrm{mc}}_{s=s_0}.$$

Let $(O'', z'' = (z_1'', z_2'', z_3''))$ be the chart on Y_r for which $g_r(z_1'', z_2'', z_3'') = (z_1'', z_1'' z_2'', z_1'' z_3'')$. Analogously as before, we obtain that the contribution to the residue at s_0 of the strict transform in Y of an open and compact subset C of $E_r \subset Y_r$ which is contained in O'' is equal to $(q-1)/(qN_r \log q)$ times

$$\left[\int_C |e|^s |\rho(0,0,0)| \chi(\mathrm{ac}\, e)\chi^l(\mathrm{ac}\, z_2)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, 1 + z_2)|z_2''|^{ls+b-1}|z_3''|^{ms+c-1}|1+z_2''|^{ns+d-1}|dz_2'' \wedge dz_3''|\right]^{\mathrm{mc}}_{s=s_0}.$$

Now we take $A = P \times P$, $B = P \times R$ and $C = R \times R$. Because these sets form a partition of $E_r \subset Y_r$, we have to prove that

$$\begin{aligned}
&\left[\int_A \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1 + z_2)|z_1|^{ks+a-1}|z_2|^{ls+b-1}|z_1+z_2|^{ns+d-1}|dz_1 \wedge dz_2|\right]^{\mathrm{mc}}_{s=s_0}\\
&+ \left[\int_B \chi^k(\mathrm{ac}\, z_1)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, z_1 + 1)|z_1'|^{ks+a-1}|z_3'|^{ms+c-1}|z_1'+1|^{ns+d-1}|dz_1' \wedge dz_3'|\right]^{\mathrm{mc}}_{s=s_0} \qquad (*)\\
&+ \left[\int_C \chi^l(\mathrm{ac}\, z_2)\chi^m(\mathrm{ac}\, z_3)\chi^n(\mathrm{ac}\, 1 + z_2)|z_2''|^{ls+b-1}|z_3''|^{ms+c-1}|1+z_2''|^{ns+d-1}|dz_2'' \wedge dz_3''|\right]^{\mathrm{mc}}_{s=s_0}
\end{aligned}$$

is equal to zero. Note that we have omitted the brackets in for example $\mathrm{ac}(z_1+z_2)$. Put $\alpha_1 = ks_0 + a$, $\alpha_2 = ls_0 + b$, $\alpha_3 = ms_0 + c$ and $\alpha_4 = ns_0 + d$. In [Se1], we proved that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. To simplify our notation, we will put $s = s_0$ in the integrand. This is not exact because the integrals do not have to converge. We actually have to calculate these integrals for complex numbers s satisfying $\mathrm{Re}(s) > 0$, and we have to evaluate the meromorphic continuation in $s = s_0$. This can be done in mind while reading the calculations. With this convention, the expression above is equal to

$$
\begin{aligned}
&\int_{P\times P}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1\wedge dz_2|\\
&\quad+\int_P\chi^k(\operatorname{ac} z_1)\chi^n(\operatorname{ac} z_1+1)|z_1|^{\alpha_1-1}|z_1+1|^{\alpha_4-1}|dz_1|\int_R\chi^m(\operatorname{ac} z_3)|z_3|^{\alpha_3-1}|dz_3|\\
&\quad+\int_R\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} 1+z_2)|z_2|^{\alpha_2-1}|1+z_2|^{\alpha_4-1}|dz_2|\int_R\chi^m(\operatorname{ac} z_3)|z_3|^{\alpha_3-1}|dz_3|.
\end{aligned}
$$

Note that

$$
H:=\int_R\chi^m(\operatorname{ac} z_3)|z_3|^{\alpha_3-1}\,|dz_3|=\begin{cases}\frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}} & \text{if } \chi^m=1\\ 0 & \text{if } \chi^m\neq 1.\end{cases}
$$

To calculate the first term in $(*)$, we partition A into

$$
\begin{aligned}
A_1 &= \{(z_1,z_2)\in P\times P\mid \operatorname{ord} z_1>\operatorname{ord} z_2\}=\bigsqcup_{i\in\mathbb{Z}_{>0}}\{(z_1,z_2)\mid \operatorname{ord} z_1>\operatorname{ord} z_2=i\}\\
A_2 &= \{(z_1,z_2)\in P\times P\mid \operatorname{ord} z_1<\operatorname{ord} z_2\}=\bigsqcup_{i\in\mathbb{Z}_{>0}}\{(z_1,z_2)\mid i=\operatorname{ord} z_1<\operatorname{ord} z_2\}\\
A_3 &= \{(z_1,z_2)\in P\times P\mid \operatorname{ord} z_1=\operatorname{ord} z_2\}=\bigsqcup_{i\in\mathbb{Z}_{>0}}\{(z_1,z_2)\mid \operatorname{ord} z_1=\operatorname{ord} z_2=i\}
\end{aligned}
$$

To calculate the third term in $(*)$, we partition C into $C_1=(R\setminus(P\cup -1+P))\times R$, $C_2=P\times R$ and $C_3=(-1+P)\times R$.

We have that N_r is a multiple of the order of χ because s_0 is a candidate pole of $Z_{f,\chi}(s)$ associated to E_r. Because $N_r=k+l+m+n$, we obtain that $1=\chi^{N_r}=\chi^{k+l+m+n}$.

Let e be the maximum of the conductors of χ^k, χ^l, χ^m and χ^n.

Case 1. $\chi^k=\chi^l=\chi^m=\chi^n=1$

The calculations for this case are the same as those for Igusa's p-adic zeta function with trivial character, and this can be found in [Se1].

Case 2. $\chi^k=\chi^m=1$ and χ^l, $\chi^n\neq 1$

Note that $\chi^l=\chi^{-n}$ and that $e=e_{\chi^l}=e_{\chi^n}$.
The contribution of A_1 to the first term in $(*)$ is equal to

$$
\begin{aligned}
&\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&=\sum_{i=1}^{\infty}q^{-i(\alpha_2+\alpha_4-2)}\int_{P^{i+1}}|z_1|^{\alpha_1-1}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|dz_2|\right)|dz_1|\\
&=-\frac{q-1}{q^2}q^{-(e-1)\alpha_1}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}+\left(\frac{q-1}{q}\right)^2\frac{q^{-e\alpha_1}}{(1-q^{-\alpha_1})(q^{\alpha_1+\alpha_2+\alpha_4-1}-1)}.
\end{aligned}
$$

For the last equality, note that by Proposition 2.5

$$\begin{aligned}\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|dz_2| &= \frac{1}{q^{i+e}}\sum_{z_2\in(R/P^e)^\times}\chi^l(z_2)\chi^n(\pi^{\mathrm{ord}\,z_1-i}\mathrm{ac}(z_1)+z_2)\\ &= \begin{cases}0 & \text{if } \mathrm{ord}\,z_1-i\in\{1,\dots,e-2\}\\ -q^{-i-1} & \text{if } \mathrm{ord}\,z_1-i=e-1\\ (q-1)q^{-i-1} & \text{if } \mathrm{ord}\,z_1-i\geq e.\end{cases}\end{aligned}$$

The contribution of A_2 to the first term in $(*)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2|\\ &= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_4-2)}\int_{P^{i+1}}\chi^l(\mathrm{ac}\,z_2)|z_2|^{\alpha_2-1}\left(\int_{P^i\setminus P^{i+1}}\chi^n(\mathrm{ac}\,z_1+z_2)|dz_1|\right)|dz_2|\\ &= 0.\end{aligned}$$

For the last equality, note that by Proposition 2.4

$$\begin{aligned}\int_{P^i\setminus P^{i+1}}\chi^n(\mathrm{ac}\,z_1+z_2)|dz_1| &= \frac{1}{q^{i+e}}\sum_{z_1\in(R/P^e)^\times}\chi^n(z_1+\pi^{\mathrm{ord}\,z_2-i}\mathrm{ac}\,z_2)\\ &= 0.\end{aligned}$$

The contribution of A_3 to the first term in $(*)$ is equal to

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^l(\mathrm{ac}\,z_2)\chi^n(\mathrm{ac}\,z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2|\\ &= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\chi^l(\mathrm{ac}\,z_2)\int_{-z_2+P^{i+1}}\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right.\\ &\qquad\left.+\chi^l(\mathrm{ac}\,z_2)\int_{(P^i\setminus P^{i+1})\setminus(-z_2+P^{i+1})}\chi^n(\mathrm{ac}\,z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_1|\right)|dz_2|\\ &= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\chi^l(\mathrm{ac}\,z_2)\int_{P^{i+1}}\chi^n(\mathrm{ac}\,z)|z|^{\alpha_4-1}|dz|\right.\\ &\qquad\left.+\chi^l(\mathrm{ac}\,z_2)q^{-i(\alpha_4-1)}\int_{(P^i\setminus P^{i+1})\setminus(z_2+P^{i+1})}\chi^n(\mathrm{ac}\,z)|dz|\right)|dz_2|\\ &= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(-\frac{\chi^l(\mathrm{ac}\,z_2)}{q^{i+e}}\sum_{z\in\mathrm{ac}\,z_2+P}\chi^n(z)\right)|dz_2|\\ &= \begin{cases}0 & \text{if } e>1\\ -\frac{q-1}{q^2}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1} & \text{if } e=1.\end{cases}\end{aligned}$$

For the third equality, note that

$$\int_{P^{i+1}} \chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz| = 0$$

and that for $z_2 \in P^i \setminus P^{i+1}$

$$\begin{aligned}
&\int_{(P^i\setminus P^{i+1})\setminus(z_2+P^{i+1})} \chi^n(\mathrm{ac}\, z)|dz| \\
&\quad = \int_{(P^i\setminus P^{i+1})} \chi^n(\mathrm{ac}\, z)|dz| - \int_{z_2+P^{i+1}} \chi^n(\mathrm{ac}\, z)|dz| \\
&\quad = \frac{1}{q^{i+e}} \sum_{z\in(R/P^e)^\times} \chi^n(z) - \frac{1}{q^{i+e}} \sum_{z\in \mathrm{ac}\, z_2+P\subset(R/P^e)^\times} \chi^n(z) \\
&\quad = -\frac{1}{q^{i+e}} \sum_{z\in \mathrm{ac}\, z_2+P\subset(R/P^e)^\times} \chi^n(z).
\end{aligned}$$

The second term of $(*)$ is equal to

$$\begin{aligned}
&\int_P \chi^n(\mathrm{ac}\, z_1{+}1)|z_1|^{\alpha_1-1}|dz_1| \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&= \left(\sum_{i=1}^{e-1} q^{-i(\alpha_1-1)} \int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1{+}1)|dz_1| + \int_{P^e} |z_1|^{\alpha_1-1}|dz_1|\right) \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&= -\frac{q-1}{q^2} q^{-(e-1)\alpha_1} \frac{1}{1-q^{-\alpha_3}} + \left(\frac{q-1}{q}\right)^2 \frac{q^{-e\alpha_1}}{(1-q^{-\alpha_1})(1-q^{-\alpha_3})}.
\end{aligned}$$

For the last equality, note that by Proposition 2.4

$$\begin{aligned}
\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1{+}1)|dz_1| &= \int_{P^i} \chi^n(\mathrm{ac}\, z_1{+}1)|dz_1| - \int_{P^{i+1}} \chi^n(\mathrm{ac}\, z_1{+}1)|dz_1| \\
&= \frac{1}{q^e} \sum_{z\in 1+P^i\subset(R/P^e)^\times} \chi^n(z) - \frac{1}{q^e} \sum_{z\in 1+P^{i+1}\subset(R/P^e)^\times} \chi^n(z) \\
&= \begin{cases} 0 & \text{if } i \in \{1,\dots,e-2\} \\ -\frac{1}{q^e} & \text{if } i = e-1. \end{cases}
\end{aligned}$$

Using Proposition 2.5 we obtain that the contribution of C_1 to the third term in $(*)$ is equal to

$$\begin{aligned}
&\int_{R\setminus(P\cup -1+P)} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1{+}z_2)|dz_2| \int_R |z_3|^{\alpha_3-1}|dz_3| \\
&\quad = \left(\frac{1}{q^e} \sum_{z_2\in(R/P^e)^\times\setminus(-1+P)} \chi^l(z_2)\chi^n(1{+}z_2)\right) \frac{q-1}{q} \frac{1}{1-q^{-\alpha_3}} \\
&\quad = \begin{cases} 0 & \text{if } e > 1 \\ -\frac{q-1}{q^2}\frac{1}{1-q^{-\alpha_3}} & \text{if } e = 1. \end{cases}
\end{aligned}$$

Using Proposition 2.5 we obtain that the contribution of C_2 to the third term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_P \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|z_2|^{\alpha_2-1}|dz_2| \\
&= \sum_{i=1}^{e-1} q^{-i(\alpha_2-1)} \int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|dz_2| + \int_{P^e} \chi^l(\mathrm{ac}\, z_2)|z_2|^{\alpha_2-1}|dz_2| \\
&= \sum_{i=1}^{e-1} \frac{q^{-i(\alpha_2-1)}}{q^{i+e}} \sum_{z\in(R/P^e)^\times} \chi^l(z)\chi^n(1+\pi^i z) \\
&= 0.
\end{aligned}$$

Using Proposition 2.5 we obtain that the contribution of C_3 to the third term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_{-1+P} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|1+z_2|^{\alpha_4-1}|dz_2| \\
&= \int_P \chi^l(\mathrm{ac}\, -1+z_2)\chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2| \\
&= \sum_{i=1}^{e-1} q^{-i(\alpha_4-1)} \int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\, -1+z_2)\chi^n(\mathrm{ac}\, z_2)|dz_2| \\
&\qquad +\chi^l(-1)\int_{P^e} \chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2| \\
&= \sum_{i=1}^{e-1} \frac{q^{-i(\alpha_4-1)}}{q^{i+e}} \sum_{z\in(R/P^e)^\times} \chi^l(-1+\pi^i z)\chi^n(z) \\
&= 0.
\end{aligned}$$

The contribution of A_1 cancels with the one of B. The contribution of A_3 cancels with the one of C_1. Here, we have to use the fact that $\alpha_1+\alpha_2+\alpha_3+\alpha_4=1$. Consequently, the contribution of $E_r \subset Y$ to the residue of $Z_{f,\chi}(s)$ at s_0 is equal to zero in this case.

Case 3. $\chi^k = \chi^l = 1$ and $\chi^m,\ \chi^n \neq 1$

Using Proposition 2.4 we obtain that the contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} |z_1|^{\alpha_1-1} \left(\int_{P^i\setminus P^{i+1}} \chi^n(\mathrm{ac}\, z_1+z_2)|dz_2| \right) |dz_1|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} |z_1|^{\alpha_1-1} \left(\frac{1}{q^{i+e}} \sum_{z_2 \in (R/P^e)^\times} \chi^n(\pi^{\operatorname{ord} z_1 - i} \operatorname{ac}(z_1) + z_2) \right) |dz_1| \\
&= 0.
\end{aligned}
$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.

Using Proposition 2.3 we obtain that the contribution of A_3 to the first term in $(*)$ is equal to

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^i \setminus P^{i+1}} \left(\int_{P^i \setminus P^{i+1}} \chi^n(\operatorname{ac} z_1 + z_2) |z_1|^{\alpha_1-1} |z_2|^{\alpha_2-1} |z_1+z_2|^{\alpha_4-1} |dz_1| \right) |dz_2| \\
&\quad = \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \int_{P^i \setminus P^{i+1}} \left(\int_{-z_2+P^{i+1}} \chi^n(\operatorname{ac} z_1 + z_2) |z_1+z_2|^{\alpha_4-1} |dz_1| \right. \\
&\qquad\qquad \left. + \int_{(P^i \setminus P^{i+1}) \setminus (-z_2+P^{i+1})} \chi^n(\operatorname{ac} z_1 + z_2) |z_1+z_2|^{\alpha_4-1} |dz_1| \right) |dz_2| \\
&\quad = \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)} \int_{P^i \setminus P^{i+1}} \int_{(P^i \setminus P^{i+1}) \setminus (-z_2+P^{i+1})} \chi^n(\operatorname{ac} z_1 + z_2) |dz_1||dz_2| \\
&\quad = \sum_{i=1}^{\infty} \frac{q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}}{q^{2(i+e)}} \sum_{z_1, z_2, z_1+z_2 \in (R/P^e)^\times} \chi^n(z_1+z_2) \\
&\quad = 0.
\end{aligned}
$$

The second and the third term in $(*)$ are both equal to zero. Indeed, we have that $H = 0$ since $\chi^m \neq 1$.

Case 4. $\chi^m = 1$ and χ^k, χ^l, $\chi^n \neq 1$

We may suppose that $e_{\chi^n} \leq e_{\chi^k}$ and that $e_{\chi^n} \leq e_{\chi^l}$. Note that $e_{\chi^k} = e_{\chi^l}$, and e is by definition this value.
The contribution of A_1 to the first term in $(*)$ is equal to

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i \setminus P^{i+1}} \chi^k(\operatorname{ac} z_1) \chi^l(\operatorname{ac} z_2) \chi^n(\operatorname{ac} z_1 + z_2) |z_1|^{\alpha_1-1} |z_2|^{\alpha_2-1} |z_1+z_2|^{\alpha_4-1} |dz_2| \right) |dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)} \int_{P^{i+1}} \chi^k(\operatorname{ac} z_1) |z_1|^{\alpha_1-1} \left(\int_{P^i \setminus P^{i+1}} \chi^l(\operatorname{ac} z_2) \chi^n(\operatorname{ac} z_1 + z_2) |dz_2| \right) |dz_1| \\
&= 0.
\end{aligned}
$$

For the last equality, note that by Proposition 2.6

$$
\int_{P^i \setminus P^{i+1}} \chi^l(\operatorname{ac} z_2) \chi^n(\operatorname{ac} z_1 + z_2) |dz_2| = \frac{1}{q^{i+e}} \sum_{z_2 \in (R/P^e)^\times} \chi^l(z_2) \chi^n(\pi^{\operatorname{ord} z_1 - i} \operatorname{ac}(z_1) + z_2)
$$

$$= 0.$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.
The contribution

$$\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.
Part 1. Using Proposition 2.7 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_1+P^{i+1}}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\Bigg(\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\ -z_1)\int_{P^{i+e}}\chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2| \\
&\qquad +\sum_{j=1}^{e-1}\int_{P^{i+j}\setminus P^{i+j+1}}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\ -z_1+z_2)\chi^n(\mathrm{ac}\, z_2)|z_2|^{\alpha_4-1}|dz_2|\Bigg)\,|dz_1| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2-2)} \\
&\qquad \int_{P^i\setminus P^{i+1}}\sum_{j=1}^{e-1}\frac{q^{-(i+j)(\alpha_4-1)}}{q^{i+j+e}}\sum_{z_2\in(R/P^e)^\times}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}(-z_1)+\pi^j z_2)\chi^n(z_2)|dz_1| \\
&= \sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2+\alpha_4-2)}\int_{P^i\setminus P^{i+1}}\sum_{j=1}^{e-1}\frac{q^{-j\alpha_4}}{q^e}\sum_{z_2\in(R/P^e)^\times}\chi^l(-1+\pi^j z_2)\chi^n(z_2)|dz_1| \\
&= \frac{q-1}{q}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{j=1}^{e-1}\frac{q^{-j\alpha_4}}{q^e}\sum_{z_2\in(R/P^e)^\times}\chi^l(-1+\pi^j z_2)\chi^n(z_2).
\end{aligned}$$

Part 2.

$$\begin{aligned}
&\sum_{i=1}^{\infty}q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})}\chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|dz_2|\right)|dz_1| \\
&= \frac{1}{q^{2e}}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(z_2)\chi^n(z_1+z_2).
\end{aligned}$$

Using Proposition 2.6 we obtain that the second term in $(*)$ is equal to H multiplied by

$$\begin{aligned}
&\int_P\chi^k(\mathrm{ac}\, z_1)\chi^n(\mathrm{ac}\, z_1+1)|z_1|^{\alpha_1-1}|dz_1| \\
&= \sum_{i=1}^{e-1}q^{-i(\alpha_1-1)}\int_{P^i\setminus P^{i+1}}\chi^k(\mathrm{ac}\, z_1)\chi^n(\mathrm{ac}\, z_1+1)|dz_1|+\int_{P^e}\chi^k(\mathrm{ac}\, z_1)|z_1|^{\alpha_1-1}|dz_1|
\end{aligned}$$

$$= \sum_{i=1}^{e-1} \frac{q^{-i(\alpha_1-1)}}{q^{i+e}} \sum_{z\in(R/P^e)^\times} \chi^k(z)\chi^n(\pi^i z+1)$$
$$= 0.$$

The contribution of C_1 to the third term in $(*)$ is equal to

$$\int_{R\setminus(P\cup -1+P)} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|dz_2| \int_R |z_3|^{\alpha_3-1}|dz_3|$$
$$= \left(\frac{1}{q^e}\sum_{z_2\in(R/P^e)^\times\setminus(-1+P)} \chi^l(z_2)\chi^n(1+z_2)\right)\frac{q-1}{q}\frac{1}{1-q^{-\alpha_3}}.$$

The contribution of C_2 to the third term in $(*)$ is 0. The calculation is the same as the calculation of the second term.
The contribution of C_3 to the third term in $(*)$ is equal to H multiplied by

$$\int_{-1+P} \chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, 1+z_2)|1+z_2|^{\alpha_4-1}|dz_1|$$
$$= \int_P \chi^l(\mathrm{ac}\ -1+z)\chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz|$$
$$= \sum_{j=1}^{e-1} q^{-j(\alpha_4-1)}\int_{P^j\setminus P^{j+1}} \chi^l(\mathrm{ac}\ -1+z)\chi^n(\mathrm{ac}\, z)|dz| + \chi^l(-1)\int_{P^e} \chi^n(\mathrm{ac}\, z)|z|^{\alpha_4-1}|dz|$$
$$= \sum_{j=1}^{e-1} \frac{q^{-j\alpha_4}}{q^e} \sum_{z\in(R/P^e)^\times} \chi^l(-1+\pi^j z)\chi^n(z)$$

The contribution of C_3 cancels with the one of the first part of A_3. Using Proposition 2.8 we obtain that the contribution of C_1 cancels with the one of the second part of A_3.

Case 5. $\chi^n = 1$ and χ^k, χ^l, $\chi^m \neq 1$

The contribution of A_1 to the first term in $(*)$ is equal to

$$\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$
$$= \sum_{i=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)}\int_{P^{i+1}} \chi^k(\mathrm{ac}\, z_1)|z_1|^{\alpha_1-1}\left(\int_{P^i\setminus P^{i+1}} \chi^l(\mathrm{ac}\, z_2)|dz_2|\right)|dz_1|$$
$$= 0.$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.
The contribution

$$\sum_{i=1}^{\infty} \int_{P^i\setminus P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.
Part 1. Using Proposition 2.9 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \int_{P^i\setminus P^{i+1}} \left(\int_{-z_1+P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \int_{P^i\setminus P^{i+1}} \left(\int_{P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, -z_1+z_2)|z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)} \sum_{j=1}^{\infty} q^{-(i+j)(\alpha_4-1)} \int_{P^i\setminus P^{i+1}} \int_{P^{i+j}\setminus P^{i+j+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, -z_1+z_2)|dz_2||dz_1| \\
&= \sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)} \sum_{j=1}^{\infty} \frac{q^{-j(\alpha_4-1)}}{q^{2i+j+2e}} \sum_{z_1,z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(-z_1+\pi^j z_2) \\
&= 0.
\end{aligned}$$

Part 2. Using Proposition 2.9 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)} \int_{P^i\setminus P^{i+1}} \left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)|dz_2|\right)|dz_1| \\
&= \frac{1}{q^{2e}} \frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1} \sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(z_2) \\
&= 0.
\end{aligned}$$

The second and the third term in $(*)$ are both equal to zero. Indeed, we have that $H=0$ since $\chi^m \neq 1$.

Case 6. χ^k, χ^l, χ^m, $\chi^n \neq 1$

Using Proposition 2.10 we obtain that the contribution of A_1 to the first term in $(*)$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{P^{i+1}} \left(\int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1| \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} q^{-i(\alpha_2+\alpha_4-2)-(i+j)(\alpha_1-1)} \\
&\qquad \int_{P^{i+j}\setminus P^{i+j+1}} \int_{P^i\setminus P^{i+1}} \chi^k(\mathrm{ac}\, z_1)\chi^l(\mathrm{ac}\, z_2)\chi^n(\mathrm{ac}\, z_1+z_2)|dz_2||dz_1| \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{q^{-i(\alpha_2+\alpha_4-2)-(i+j)(\alpha_1-1)}}{q^{2i+j+2e}} \sum_{z_1,z_2\in(R/P^e)^\times} \chi^k(z_1)\chi^l(z_2)\chi^n(\pi^j z_1+z_2) \\
&= 0.
\end{aligned}$$

Analogously, we obtain that the contribution of A_2 to the first term in $(*)$ is equal to 0.

The contribution

$$\sum_{i=1}^{\infty}\int_{P^i\setminus P^{i+1}}\left(\int_{P^i\setminus P^{i+1}}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|z_1|^{\alpha_1-1}|z_2|^{\alpha_2-1}|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|$$

of A_3 to the first term in $(*)$ is the sum of two parts.
Part 1. Using Proposition 2.10 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{-z_1+P^{i+1}}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|z_1+z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\int_{P^i\setminus P^{i+1}}\left(\int_{P^{i+1}}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac}\ -z_1+z_2)\chi^n(\operatorname{ac} z_2)|z_2|^{\alpha_4-1}|dz_2|\right)|dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2-2)}\sum_{j=1}^{\infty} q^{-(i+j)(\alpha_4-1)}\\
&\qquad\qquad\int_{P^i\setminus P^{i+1}}\int_{P^{i+j}\setminus P^{i+j+1}}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac}\ -z_1+z_2)\chi^n(\operatorname{ac} z_2)|dz_2||dz_1|\\
&\quad=\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\sum_{j=1}^{\infty}\frac{q^{-j(\alpha_4-1)}}{q^{2i+j+2e}}\sum_{z_1,z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(-z_1+\pi^j z_2)\chi^n(z_2)\\
&\quad=0.
\end{aligned}$$

Part 2. Using Proposition 2.10 we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} q^{-i(\alpha_1+\alpha_2+\alpha_4-3)}\int_{P^i\setminus P^{i+1}}\left(\int_{(P^i\setminus P^{i+1})\setminus(-z_1+P^{i+1})}\chi^k(\operatorname{ac} z_1)\chi^l(\operatorname{ac} z_2)\chi^n(\operatorname{ac} z_1+z_2)|dz_2|\right)|dz_1|\\
&\quad=\frac{1}{q^{2e}}\frac{1}{q^{\alpha_1+\alpha_2+\alpha_4-1}-1}\sum_{z_1,z_2,z_1+z_2\in(R/P^e)^\times}\chi^k(z_1)\chi^l(z_2)\chi^n(z_1+z_2)\\
&\quad=0.
\end{aligned}$$

The second and the third term in $(*)$ are again both equal to zero.

We have treated all the cases by using the geometric symmetry of the problem, so our proof is finished. □

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