A round table problem

For a business meeting n persons are sitting around a round table. Unfortunately, they didn't notice that nameplates were placed. We wonder whether the table can be turned in such a way that at least two persons are sitting in front of their nameplate.

This might be impossible when n is odd. Identify the n persons with the integers modulo n such that $a-1$ and $a+1$ are the neighbours of a. For this identification it is necessary and sufficient to call an arbitrary person 0 and to choose a direction. Because 2 is invertible modulo the odd number n, $2a_1 = 2a_2$ implies $a_1 = a_2$. Consequently, it is possible that the nameplate of a is in front of $2a$ for every a. By construction every a is sitting in front of his nameplate after a rotation of the people over a persons in the chosen direction. We will always rotate the people instead of the table for the sake of elegancy.

We prove by contradiction that it is possible when n is even. Suppose that the table cannot be turned in such a way that at least two persons are sitting in front of their nameplate. After every rotation there is thus exactly one person sitting right. Consider the permutation π defined by $\pi(i) = j$ if and only if the nameplate of i is in front of j. Identify the rotations with the integers modulo n by calling a rotation over a persons in the chosen direction simply a. Then we have

$$\sum_{i=0}^{n-1} i \text{ modulo } n = \sum_{i=0}^{n-1} (\pi(i) - i) \text{ modulo } n,$$

because in both sides we add all the rotations in the rotation group. Remark that the index i in the first sum runs over the rotations, while the index i in the second sum runs over the persons. The right hand side of this equation is equal to $\sum_{i=0}^{n-1} \pi(i)$ modulo n minus $\sum_{i=0}^{n-1} i$ modulo n and consequently zero because π is a permutation. But the left hand side of the equation is equal to $n(n-1)/2$ modulo n, which is different from zero. A contradiction.

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