VANISHING OF PRINCIPAL VALUE INTEGRALS ON SURFACES

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Abstract. Principal value integrals are associated to multi–valued rational differential forms with normal crossings support on a non–singular algebraic variety. We prove their vanishing on rational surfaces in the context of a conjecture of Denef–Jacobs. As an application we obtain a strong vanishing result for candidate poles of p–adic and motivic Igusa zeta functions.

Introduction

0.1. Real and p–adic principal value integrals were first introduced by Langlands in the study of orbital integrals [Lan1] [Lan2] [LS1] [LS2]. They are associated to multi–valued differential forms on real and p–adic manifolds, respectively.

Let for instance X be a non–singular projective algebraic variety of dimension n over $\mathbb{Q}_p$ (the field of p–adic numbers). Denoting by Ω^n_X the vector space of *rational* differential n–forms on X, take $\omega \in (\Omega^n_X)^{\otimes d}$ defined over $\mathbb{Q}_p$; we then write formally $\omega^{1/d}$ and consider it as a multi–valued rational differential form on X.

We suppose that the support $|\operatorname{div}\omega|$ of $\operatorname{div}\omega$ has normal crossings (over $\mathbb{Q}_p$) on X; say $D_i, i \in S$, are its irreducible components. Let $\operatorname{div}\omega^{1/d} := \frac{1}{d}\operatorname{div}\omega = \sum_{i\in S}(\alpha_i - 1)D_i$, where then the $\alpha_i \in \frac{1}{d}\mathbb{Z}$. If $\omega^{1/d}$ has *no logarithmic poles*, i.e. if all $\alpha_i \neq 0$, the *principal value integral* $PV \int_{X(\mathbb{Q}_p)} |\omega^{1/d}|_p$ of $\omega^{1/d}$ on $X(\mathbb{Q}_p)$ is defined as follows. Cover $X(\mathbb{Q}_p)$ by (disjoint) small enough open balls B on which there exist local coordinates $x_1, \dots, x_n$ such that all D_i are coordinate hyperplanes. Consider for each B the converging integral $\int_B |x_1 x_2 \cdots x_n|_p^s |\omega^{1/d}|_p$ for $s \in \mathbb{C}$ with $\mathcal{R}(s) >> 0$, take its meromorphic continuation to $\mathbb{C}$ and evaluate this in $s = 0$; then add all these contributions. One can check that the result is independent of all choices.

In the real setting we proceed similarly but then we also need a partition of unity, and we have to assume that $\omega^{1/d}$ has no integral poles, i.e. the $\alpha_i \notin \mathbb{Z}_{\leq 0}$. Here the independency result is somewhat more complicated; it was verified in detail in [Ja1].

0.2. Denef and Jacobs proved a vanishing result for real principal value integrals, and conjectured a similar statement in the p–adic case. In both cases let $\mathcal{L}(\omega^{1/d})$ be the locally constant sheaf of $\mathbb{C}$–vector spaces on $X \setminus |\operatorname{div}\omega|$ associated to $\omega^{1/d}$. It has rank 1, a non–zero section on a connected open being an analytic branch of $\omega^{1/d}$ multiplied with a complex number. (In the p–adic case we choose an embedding of $\mathbb{Q}_p$ into $\mathbb{C}$.)

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Theorem [DJ, 1.1.4][Ja1]**.** *Let X be a non–singular projective algebraic variety of (complex) dimension n, defined over $\mathbb{R}$. If $H^n(X(\mathbb{C}) \setminus |\operatorname{div}\omega|, \mathcal{L}(\omega^{1/d})) = 0$, then $PV \int_{X(\mathbb{R})} |\omega^{1/d}| = 0$.*

Conjecture [DJ, 1.2.2]**.** *Let X be a non-singular projective algebraic variety, defined over $\mathbb{Q}_p$. If $H^i(X(\mathbb{C}) \setminus |\operatorname{div}\omega|, \mathcal{L}(\omega^{1/d})) = 0$ for all $i \geq 0$, then $PV \int_{X(\mathbb{Q}_p)} |\omega^{1/d}|_p = 0$.*

(The authors are cautious and mention that perhaps one has to suppose also some good reduction mod p and that all $\alpha_i \notin \mathbb{Z}$.)

0.3. In [Ve7] we 'upgraded' p–adic principal value integrals to motivic ones, in the same spirit as how motivic integration and motivic zeta functions were inspired by (usual) p–adic integration and p–adic Igusa zeta functions. See [DL2][DL3] or the surveys [DL4][Loo][Ve6] for these notions.

More precisely, let X be a non–singular algebraic variety (say over $\mathbb{C}$) of dimension n and $\omega^{1/d}$ a multi–valued differential form on X. Let as above $\operatorname{div}(\omega^{1/d}) = \sum_{i\in S}(\alpha_i - 1)D_i$ be a normal crossings divisor. We denote also $D_I^\circ := (\cap_{i\in I} D_i) \setminus (\cup_{\ell\notin I} D_\ell)$ for $I \subset S$; so $X = \coprod_{I\subset S} D_I^\circ$. If $\omega^{1/d}$ has no logarithmic poles, then the *motivic principal value integral* of $\omega^{1/d}$ on X is

$$PV \int_X \omega^{1/d} = L^{-n} \sum_{I\subset S} [D_I^\circ] \prod_{i\in I} \frac{L-1}{L^{\alpha_i} - 1}.$$

Here $[\cdot]$ denotes the class of a variety in the Grothendieck ring of algebraic varieties, and $L := [\mathbb{A}^1]$, see (1.1). We refer to [Ve7, §2] for a motivation for this expression. Note for instance that this is just the formula for the *converging* motivic integral associated to the $\mathbb{Q}$–divisor $\operatorname{div}(\omega^{1/d})$ if all $\alpha_i > 0$.

It is natural to 'upgrade' also the conjecture in (0.2) to this setting (maybe also assuming that all $\alpha_i \notin \mathbb{Z}$) : *if $H^i(X(\mathbb{C}) \setminus |\operatorname{div}\omega|, \mathcal{L}(\omega^{1/d})) = 0$ for all $i \geq 0$, then $PV \int_X \omega^{1/d} = 0$.*

0.4. In this paper we attack (the motivic version of) the conjecture for surfaces, more precisely *rational* surfaces.

We note that, in dimension 2, this is the crucial class to study : there is no 'classification' for configurations $|\operatorname{div}\omega| \subset X$ with all $H^i(X \setminus |\operatorname{div}\omega|, \mathcal{L}(\omega^{1/d})) = 0$ on a rational surface X. For non–rational surfaces it is conceivable that the conjecture can be approached through the classification of such configurations from [GP] and [Ve3]. We plan to report on this later. (Also for the applications that we will prove here, the class of rational surfaces is the crucial one, see (0.7).)

Our main theorem is as follows.

Theorem. *Let X be a non–singular projective rational surface and $\omega^{1/d}$ a multi–valued differential form on X without logarithmic poles (in particular $|\operatorname{div}\omega|$ has normal crossings).*

(1) Suppose that $B := |\operatorname{div}\omega|$ is connected. *If $\chi(X \setminus B) \leq 0$, then $PV \int_X \omega^{1/d} = 0$.*

(2) More generally, let B be any connected *normal crossings divisor satisfying $B \supset |\operatorname{div}\omega|$. If $\chi(X \setminus B) \leq 0$, then $PV \int_X \omega^{1/d} = 0$.*

Statement (1) is somewhat weaker than the conjecture because of the connectivity condition. On the other hand it is clearly stronger since we assume only that $\chi(X \setminus B) \leq 0$, instead of the vanishing of all $H^i(X \setminus B, \mathcal{L}(\omega^{1/d}))$. (And we do not need the extra assumption $\alpha_i \notin \mathbb{Z}$.) The generalization (2) is important in view of the applications on motivic zeta functions and is natural in that context, see (0.8).

The important ingredients in our proof are the structure theorem of [Ve3] for such configurations $B \subset X$ with $\chi(X \setminus B) \leq 0$, and the notion of a more general principal value integral when $\omega^{1/d}$ is allowed to have 'some' logarithmic poles, see (1.4).

0.5. Real and p–adic principal value integrals appear as coefficients of asymptotic expansions of oscillating integrals and fibre integrals, and as residues of poles of distributions $|f|^\lambda$ and p–adic Igusa zeta functions, respectively. See [Ja2, §1] for an overview and [AVG][De2][Ig1][Ig2][Ja3][Lae] for more details.

We have in particular that the cancelation of some given candidate pole of a p–adic Igusa zeta function is essentially equivalent to the vanishing of an associated p–adic principal value integral. Here we are interested in the analogous phenomenon for poles of the motivic zeta function versus associated motivic principal value integrals.

0.6. Denef and Loeser [DL2] associated to a non–constant regular function f on a non–singular algebraic variety M of dimension $n+1$ its motivic zeta function $\mathcal{Z}_{\text{mot}}(f;T)$; here T is a variable. They obtained the following formula for it in terms of an embedded resolution $h : Y \to M$ of the hypersurface $\{f = 0\}$. Let $E_j, j \in K$, be the irreducible components of $h^{-1}\{f = 0\}$ and N_j and $\nu_j - 1$ the multiplicity of E_j in the divisors $\operatorname{div}(f \circ h)$ and $\operatorname{div}(h^* dx)$, respectively, where dx is a local generator of the sheaf of $(n+1)$–forms on M. Denote $E_J^\circ := (\cap_{j \in J} E_j) \setminus (\cup_{\ell \notin J} E_\ell)$ for $J \subset K$. Then

$$\mathcal{Z}_{\text{mot}}(f;T) = L^{-(n+1)} \sum_{J \subset K} [E_J^\circ] \prod_{j \in J} \frac{(L-1)T^{N_j}}{L^{\nu_j} - T^{N_j}}\,.$$

Fix a 'candidate pole' L^{ν_j/N_j} of $\mathcal{Z}_{\text{mot}}(f;T)$; see (3.4) for more explanations. In the generic situation that $\nu_j/N_j \neq \nu_i/N_i$ for all $i \neq j$, the cancelation of the candidate pole L^{ν_j/N_j} is equivalent to the vanishing of 'its residue'

$$L^{-n} \sum_{I \subset S_j} [D_I^\circ] \prod_{i \in I} \frac{L-1}{L^{\alpha_i} - 1},$$

where $S_j = \{i \in K \setminus \{j\} \mid E_i \text{ intersects } E_j\}$, $D_i := E_j \cap E_i$ and $\alpha_i = \nu_i - (\nu_j/N_j)N_i$ for $i \in S_j$, and D_I° is as usual. This looks like a motivic principal value integral on E_j, and indeed it is equal to $PV \int_{E_j} \omega^{1/d}$, where $\omega^{1/d}$ is some Poincaré residue; see (3.5).

Suppose now that E_j is mapped to a point by h. Denote $B_j := \cup_{i \in S_j} D_i$. The famous Monodromy Conjecture predicts more or less that (generically), if $(-1)^n \chi(E_j \setminus B_j) \leq 0$, then L^{ν_j/N_j} is no pole of $Z_{\text{mot}}(f;T)$, see (3.5).

0.7. When $n = 2$, we now can prove this expected cancelation of candidate poles as a consequence of our Main Theorem. More precisely, we may suppose that h is a composition of blowing–ups as in [Hi]. Then the exceptional surface E_j is created during the resolution process h either as a projective plane by blowing up a point, or as a ruled surface by blowing up a non–singular curve. Then, with B_j as above, we can show as corollary of (0.4):

Theorem. *Let $\chi(E_j \setminus B_j) \leq 0$.*

(1) If E_j is created by blowing up a point, then we have always *that L^{ν_j/N_j} is no pole.*

(2) If E_j is created by blowing up a rational curve, then L^{ν_j/N_j} is no pole whenever B_j is connected.

In fact we obtain a somewhat stronger statement; see (0.8(i)) below. But first we want to comment on this new result.

Concerning (1), the best general result up to now [Ve2] was the analogous cancelation in the context of p–adic Igusa zeta functions when the centre of the blowing up is a point of multiplicity at most 4 on the strict transform of $\{f = 0\}$. This was achieved using a lengthy classification of all possible configurations with $\chi(E_j \setminus B_j) \leq 0$ under this multiplicity restriction.

Concerning (2), it is important to note that non–connected intersection configurations on E_j are very rare, see (3.7). And moreover, the case of *rational* centres is the crucial one. When E_j is created by blowing up a curve of genus $g \geq 2$ we already proved the expected cancelation in [Ve2], and we can now also handle the case $g = 1$ completely by combining [Ve2] and [Ve3] with recent work of Rodrigues [Ro2].

0.8. *Remarks.* (i) In fact we assume only that $\nu_j/N_j \neq \nu_i/N_i$ for the $i \in S_j$, and we show that 'the contribution of E_j to the residue of L^{ν_j/N_j}' vanishes when expected.

(ii) Similar vanishing results hold in the context of much more general motivic zeta functions, for instance those of [Ve4] associated to an effective $\mathbb{Q}$–Cartier divisor on any $\mathbb{Q}$–Gorenstein threefold (instead of just a hypersurface on $\mathbb{A}^3$). On the other hand, (0.7) specializes to the context of Hodge, topological and p–adic zeta functions.

(iii) Note that in (0.6) it is possible that some $\alpha_i = 1$, and thus that $B_j (:= \cup_{i \in S_j} D_i) \supsetneq |\operatorname{div} \omega^{1/d}|$. So the more general setting of part (2) of our main theorem pops up naturally in this context of poles of zeta functions.

0.9. We will work over the base field $\mathbb{C}$ of complex numbers. In §1 we introduce the more general principal value integrals on surfaces, allowing 'some' logarithmic poles, which we need for the proof of the main theorem in §2. Then in §3 we obtain as a corollary the cancelation of candidate poles for zeta functions.

Acknowledgement. We would like to remark that [ACLM1, §2] contributed to our inspiration for the proof of the main theorem. And we thank Bart Rodrigues for his useful remarks and suggestions.

1. Generalized principal value integrals on surfaces

1.1. We first recall briefly the notion of Grothendieck ring of varieties and related constructions.

(i) The Grothendieck ring $K_0(\mathrm{Var})$ of complex algebraic varieties is the free abelian group generated by the symbols $[V]$, where V is a variety, subject to the relations $[V] = [V']$ if V is isomorphic to V', and $[V] = [V \setminus W] + [W]$ if W is closed in V. Its ring structure is given by $[V] \cdot [W] := [V \times W]$. (This ring is quite mysterious; see [Po] for the recent proof that it is not a domain.) Usually, one abbreviates $L := [\mathbb{A}^1]$.

For the sequel we need to extend $K_0(\mathrm{Var})$ with fractional powers of L and to localize. Fix $d \in \mathbb{Z}_{>0}$; we consider

$$K_0(\mathrm{Var})[L^{-1/d}] := \frac{K_0(\mathrm{Var})[T]}{(LT^d - 1)}$$

(where $L^{-1/d} := \bar{T}$). We then localize this ring with respect to the elements $L^{i/d} - 1$, $i \in \mathbb{Z} \setminus \{0\}$. What we really need is the subring of this localization generated by $K_0(\mathrm{Var})$, L^{-1} and the elements $(L-1)/(L^{i/d}-1), i \in \mathbb{Z} \setminus \{0\}$; we denote this subring by $\mathcal{R}_d$. (We do not know whether or not $\mathcal{R}_d$ has zero divisors.)

(ii) For a variety V, we denote by $h^{p,q}(H^i_c(V,\mathbb{C}))$ the rank of the (p,q)–Hodge component in the mixed Hodge structure of the ith cohomology group with compact support of V. The *Hodge polynomial* of V is

$$H(V) = H(V;u,v) := \sum_{p,q} \big(\sum_{i \geq 0} (-1)^i h^{p,q}(H^i_c(V,\mathbb{C}))\big) u^p v^q \in \mathbb{Z}[u,v].$$

Precisely by the defining relations of $K_0(\mathrm{Var})$, there is a well–defined ring homomorphism $H : K_0(\mathrm{Var}) \to \mathbb{Z}[u,v]$, determined by $[V] \mapsto H(V)$. It induces a ring homomorphism H from $\mathcal{R}_d$ to the 'rational functions in u,v with fractional powers'.

(iii) The topological Euler characteristic $\chi(V)$ of a variety V satisfies $\chi(V) = H(V;1,1)$ and we obtain a ring homomorphism $\chi : K_0(\mathrm{Var}) \to \mathbb{Z}$, determined by $[V] \mapsto \chi(V)$. Since $\chi(L) = 1$, it induces a ring homomorphism $\chi : \mathcal{R}_d \to \mathbb{Q}$ by declaring $\chi((L-1)/(L^{i/d} - 1)) = i/d$ (see e.g. [DL2][Ve4] for similar constructions).

1.2. On surfaces we want to extend the notion of principal value integral of (0.3) in two ways. First we will allow the differential form $\omega^{1/d}$ to have 'some' logarithmic poles. Another somewhat technical generalization consists in considering a normal crossings divisor whose support *contains* $|\operatorname{div}\omega|$. In view of the application on candidate poles of zeta functions, this is natural; see (0.8(iii)) or (3.5). More precisely we fix the following setting.

1.3. Let X be a non–singular projective surface, $\omega^{1/d}$ a multi–valued differential form on X, and $D = \cup_{i\in T} C_i$ a normal crossings divisor on X satisfying $D \supset |\operatorname{div}\omega|$. We write formally $\operatorname{div}\omega^{1/d} = \sum_{i\in T}(\alpha_i - 1)C_i$, where thus $\alpha_i = 1$ if $C_i \not\subset |\operatorname{div}\omega|$. We put the following restriction on the possible logarithmic poles of $\omega^{1/d}$.
If for $i \in T$ we have $\alpha_i = 0$, then

(1) C_i is rational,

(2) no C_ℓ that intersects C_i has $\alpha_\ell = 0$,

(3) in all but at most two intersection points of C_i with other C_ℓ the intersecting curve C_ℓ has $\alpha_\ell = 1$.

We call such a pair $(D, \omega^{1/d})$ *allowed.*

When $\alpha_i = 0$ the adjunction formula $(K_X + C_i) \cdot C_i = K_{C_i}$ and the restrictions above easily yield that either two curves C_{i_1} and C_{i_2} with $\alpha_{i_1} \neq 1 \neq \alpha_{i_2}$ intersect C_i and then $\alpha_{i_1} + \alpha_{i_2} = 0$, or only one curve C_{i_1} with $\alpha_{i_1} \neq 1$ intersects C_i and then $\alpha_{i_1} = -1$. (Here and further on, when applying the adjunction formula we always consider $\operatorname{div} \omega^{1/d}$ as a representative of K_X.)

1.4. Definition. Let X be a non–singular projective surface and $(D, \omega^{1/d})$ an allowed pair on X. We write $\operatorname{div} \omega^{1/d} = \sum_{i \in T} (\alpha_i - 1) C_i$ as in (1.3) and $C_I^\circ := (\cap_{i \in I} C_i) \setminus (\cup_{\ell \notin I} C_\ell)$ for $I \subset T$ as before. Furthermore for $i \in T$ with $\alpha_i = 0$ we denote by $C_i \cdot C_i$ the self–intersection number of C_i and by $C_j, j \in T_i (\subset T)$, the curves that intersect C_i. To $(D, \omega^{1/d})$ we associate the invariant

$$\mathcal{E}_X(D, \omega^{1/d}) := \sum_{\substack{I \subset T \\ \forall i \in I : \alpha_i \neq 0}} [C_I^\circ] \prod_{i \in I} \frac{L-1}{L^{\alpha_i} - 1} + \sum_{\substack{i \in T \\ \alpha_i = 0}} (-C_i \cdot C_i) \prod_{j \in T_i} \frac{L-1}{L^{\alpha_j} - 1},$$

living in $\mathcal{R}_d$. In the last sum the expression $(-C_i \cdot C_i) \prod_{j \in T_i} \frac{L-1}{L^{\alpha_j}-1}$ is the easiest to write down 'uniformly', but, since at most two of the α_j are different from 1, this expression boils down to the following.

(1) If C_i intersects two curves C_{i_1} and C_{i_2} with $\alpha_{i_1} \neq 1 \neq \alpha_{i_2}$, we get

$$(-C_i \cdot C_i) \frac{(L-1)^2}{(L^{\alpha_{i_1}} - 1)(L^{\alpha_{i_2}} - 1)} = (C_i \cdot C_i) \frac{(L-1)^2 L^{\alpha_{i_1}}}{(L^{\alpha_{i_1}} - 1)^2} = (C_i \cdot C_i) \frac{(L-1)^2 L^{\alpha_{i_2}}}{(L^{\alpha_{i_2}} - 1)^2}.$$

(2) If C_i intersects only one curve C_{i_1} with $\alpha_{i_1} \neq 1$, we get

$$(-C_i \cdot C_i) \frac{L-1}{L^{\alpha_{i_1}} - 1} = (C_i \cdot C_i) L.$$

Note. (i) In fact the C_i with $\alpha_i = 1$, i.e. those C_i not belonging to $|\operatorname{div} \omega|$, play no role in the definition of $\mathcal{E}_X(D, \omega^{1/d})$: we could as well consider instead of T only $\{i \in T \mid C_i \subset |\operatorname{div} \omega|\}$. So this invariant is really an invariant of $\omega^{1/d}$ only. However, for the sequel it is useful to introduce it as above.

(ii) As a motivation for the expression for the contribution of C_i with $\alpha_i = 0$: it is a kind of limit of the 'total contribution of C_i' in the formula of (0.3) for $PV \int_X \omega^{1/d}$ if $\alpha_i \neq 0$. See e.g. [Ve5, 3.3].

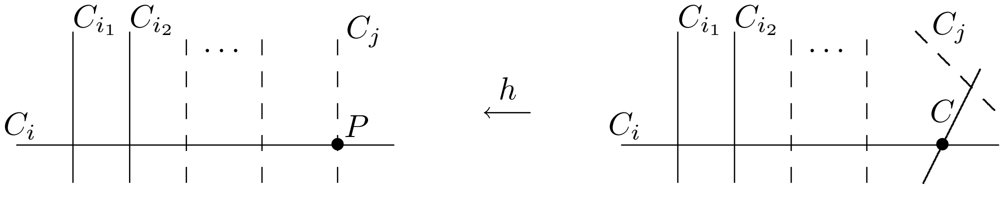


FIGURE 1

1.5. Lemma. *Let X be a non–singular projective surface, $P \in X$ and $h : \tilde{X} \to X$ the blowing–up of X with centre P. Let $(D, \omega^{1/d})$ be allowed on X, writing as usual* $\operatorname{div} \omega^{1/d} = \sum_{\ell \in T}(\alpha_\ell - 1)C_\ell$. *Then*

(1) $(h^{-1}D, h^*\omega^{1/d})$ is allowed on $\tilde{X}$,

(2) $\mathcal{E}_{\tilde{X}}(h^{-1}D, h^*\omega^{1/d}) = \mathcal{E}_X(D, \omega^{1/d})$ except when (on X) there exists a curve C_i with $\alpha_i = 0$ and curves C_{i_1} and C_{i_2}, intersecting C_i, with $\alpha_{i_1} + \alpha_{i_2} = 0$ and $\{\alpha_{i_1}, \alpha_{i_2}\} \neq \{-1, 1\}$, such that $P \in C_i$, $P \notin C_{i_1}$ and $P \notin C_{i_2}$. (And in this exceptional case we do have inequality.)

Proof. The necessary (easy) computations are essentially in [ACLM1] and [Ve5, 3.5]. We just illustrate the exceptional case, see Figure 1. Note that in this case, since $(D, \omega^{1/d})$ is allowed, if there is another curve C_j passing through P, it must satisfy $\alpha_j = 1$. And by Note (1.4(i)) we may as well assume that only C_{i_1} and C_{i_2} intersect C_i.

Let C denote the exceptional curve of h. Then $h^{-1}D$ consists of the union of C and the strict transforms of the C_ℓ. And since $\alpha_i = 0$ we have that C does not appear in the divisor of $h^*\omega^{1/d}$. So we write formally $\operatorname{div}(h^*\omega^{1/d}) = \sum_{\ell \in T}(\alpha_\ell - 1)C_\ell + (\alpha - 1)C$ with $\alpha = 1$. We have to compare the contributions of C_i to $\mathcal{E}_X(D, \omega^{1/d})$ and of $C_i \cup C$ to $\mathcal{E}_{\tilde{X}}(h^{-1}D, h^*\omega^{1/d})$. These are

$$-(C_i \cdot C_i)\frac{(L-1)^2}{(L^{\alpha_{i_1}} - 1)(L^{\alpha_{i_2}} - 1)}$$

and

$$-(C_i \cdot C_i - 1)\frac{(L-1)^2}{(L^{\alpha_{i_1}} - 1)(L^{\alpha_{i_2}} - 1)} + L,$$

respectively. Their difference $\frac{(L-1)^2}{(L^{\alpha_{i_1}}-1)(L^{\alpha_{i_2}}-1)} + L$ is nonzero (in $\mathcal{R}_d$). (On the other hand, when $\{\alpha_{i_1}, \alpha_{i_2}\} = \{-1, 1\}$, this difference would be $-L + L = 0$.) □

2. Rational surfaces

2.1. We first summarize the structure theorem of [Ve3] and some of its refinements, which will be the starting point of the proof of our main theorem. Remember that a non–singular rational curve with self–intersection -1 is called a (-1)–curve.

Structure Theorem. *Let X be a non–singular projective rational surface. Let B be a* connected *normal crossings curve on X with $\chi(X \setminus B) \leq 0$. Assume that X does not contain any (-1)–curve disjoint from B.*

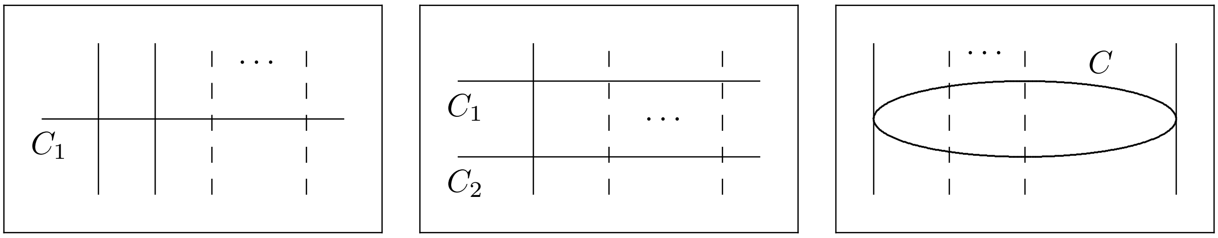
By [GP, Theorem 3] there is a dominant morphism $\varphi : X \setminus B \to \mathbb{P}^1$; let $h : \tilde{X} \to X$ be the minimal *morphism that resolves the indeterminacies of φ, considered as rational map from X to $\mathbb{P}^1$.*

(1) Then there exists a connected curve $B' \supset B$ with $\chi(X \setminus B') \leq \chi(X \setminus B) \leq 0$, such that the morphism $\varphi \circ h$ decomposes as

$$\tilde{X} \xrightarrow{g} \Sigma \xrightarrow{\pi} \mathbb{P}^1,$$

where g is a composition of blowing–downs with exceptional curve in $h^{-1}B'$, and $\pi : \Sigma \to \mathbb{P}^1$ is a ruled surface; see Diagram 1. Moreover, $h^{-1}B'$ has normal crossings in $\tilde{X}$.

(2) We can require the configuration $g(h^{-1}B') \subset \Sigma$ to be one of the configurations in Figure 2.



(a) *(b)* *(c)*

Figure 2

Here C_1 and C_2 are sections of π, C is a non–singular curve for which $\pi|_C : C \to \mathbb{P}^1$ has degree 2 (a 'bisection'), and the other curves are fibres of π. The minimal number of fibres in (a) and (b) is 2 and 1, respectively; in (c) there must pass a fibre through each ramification point of $\pi|_C$, and we can have any number of other fibres.

(3) Irreducible curves $\mathcal{C} \subset h^{-1}B'$, which are not components of $h^{-1}B$, occur only in fibres of g. Moreover, any fibre of g contains at most one *such curve $\mathcal{C}$ and*

$$\begin{cases} \operatorname{card}(\mathcal{C} \cap h^{-1}B) = 1 & \textit{if} \quad \chi(\Sigma \setminus g(h^{-1}B')) < 0 \\ \operatorname{card}(\mathcal{C} \cap h^{-1}B) = 2 & \textit{if} \quad \chi(\Sigma \setminus g(h^{-1}B')) = 0. \end{cases}$$

Note that always $\chi(\Sigma \setminus g(h^{-1}B')) = 0$ in cases (b) and (c).

Proof. Combine essentially (3.3), (3.5) and (4.3) in [Ve3]. $\square$

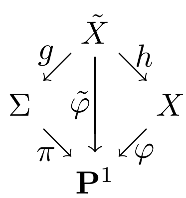


Diagram 1

2.2. We will denote by $p : \tilde{\Sigma} \to \Sigma$ the minimal embedded resolution of the configuration $D := g(h^{-1}B') \subset \Sigma$ in case (c) of the structure theorem. When $\omega^{1/d}$ is a multi–valued differential form on Σ with $D \supset |\operatorname{div}\omega|$, we will slightly abuse the terminology of (1.3) and say that $(D, \omega^{1/d})$ is allowed on Σ if $(p^{-1}D, p^*\omega^{1/d})$ is allowed on $\tilde{\Sigma}$. In that case we also put $\mathcal{E}_\Sigma(D, \omega^{1/d}) := \mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d})$.

Lemma. *Let $D := g(h^{-1}B') \subset \Sigma$ in the structure theorem and let $\omega^{1/d}$ be a multi-valued differential form on Σ. Assume in case (c) that C is rational (this is equivalent to $\pi|_C$ having exactly two ramification points). If $(D, \omega^{1/d})$ is allowed on Σ, then $\mathcal{E}_\Sigma(D, \omega^{1/d}) = 0$.*

Proof. We write as usual $D = \cup_{i\in T} C_i$ and $\operatorname{div}\omega^{1/d} = \sum_{i\in T}(\alpha_i - 1)C_i$. Applying the adjunction formula to a generic fibre of π yields in cases (a), (b) and (c) that $\alpha_1 = -1$, $\alpha_1 + \alpha_2 = 0$ and $\alpha = 0$, respectively. We first treat the cases (a) and (b).

If no $\alpha_i = 0$ this is well known and easily verified; see e.g. [Ve2] for a similar computation. The point is that the contribution of *any* fibre of π to $\mathcal{E}_X(D, \omega^{1/d})$ is zero. Now if $\alpha_i = 0$ for some fibre C_i of π, the contribution of C_i is still zero, simply because its self–intersection $C_i \cdot C_i = 0$. This finishes already case (a).

In case (b) we are left with the following possibility: $\alpha_1 = \alpha_2 = 0$ and (omitting the possible fibres C_ℓ with $\alpha_\ell = 1$) there is either only one fibre $C_i \subset D$ with then necessarily $\alpha_i = -1$, or there are two fibres C_i and C_i' in D with $\alpha_i + \alpha_i' = 0$ and $\alpha_i \neq 0 \neq \alpha_i'$. We compute $\mathcal{E}_X(D, \omega^{1/d})$ in this last case:

$$\begin{aligned}\mathcal{E}_X(D, \omega^{1/d}) =& (L-1)^2 + (L-1)\frac{L-1}{L^{\alpha_i}-1} + (L-1)\frac{L-1}{L^{\alpha_i'}-1} \\ &+ (-C_1 \cdot C_1)\frac{(L-1)^2}{(L^{\alpha_i}-1)(L^{\alpha_i'}-1)} + (-C_2 \cdot C_2)\frac{(L-1)^2}{(L^{\alpha_i}-1)(L^{\alpha_i'}-1)}.\end{aligned}$$

Since $C_1 \cdot C_1 = -C_2 \cdot C_2$ (see e.g. [Ha, Theorem V.2.17]) the last two terms cancel and, since $\alpha_i + \alpha_i' = 0$, we obtain

$$\mathcal{E}_X(D, \omega^{1/d}) = (L-1)^2 \frac{L^{\alpha_i + \alpha_i'} - 1}{(L^{\alpha_i}-1)(L^{\alpha_i'}-1)} = 0.$$

For case (c) we have to consider $\mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d})$. We denote the two curves in $p^{-1}D$ which intersect $C(\subset \tilde{\Sigma})$ in P_1 and P_2 by C_1 and C_2, respectively. See Figure 3.

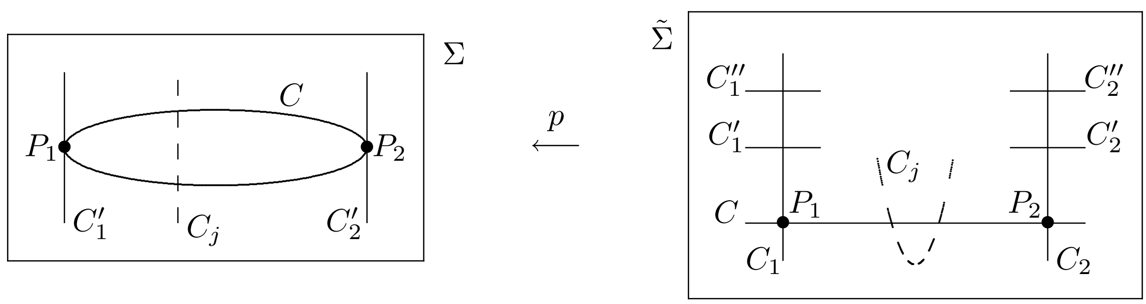


FIGURE 3

Claim. *A fibre C_j in Σ not containing P_1 or P_2 must have $\alpha_j = 1$.* Indeed, by allowedness, if $\alpha_j \neq 1$ we must have $\alpha_1 = \alpha_2 = 1$ and also $\alpha_\ell = 1$ for possible other fibres C_ℓ on Σ. But then the adjunction formula for C on $\tilde{\Sigma}$ yields $\alpha_j = 0$, contradicting the other requirement for allowedness.

Consequently, for the computation of $\mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d})$ we can neglect possible fibres not containing P_1 or P_2, and we know that $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 \neq 0 \neq \alpha_2$. The adjunction formula on $\tilde{\Sigma}$ for C_i' and C_i'' yields $\alpha_i' = \alpha_i'' = \frac{\alpha_i+1}{2}$ for $i = 1, 2$. Using the structure of $\operatorname{Pic}\Sigma$ one computes that the self–intersection $C \cdot C = 4$ on Σ, see e.g. [Ve1, Remark 6.7]. Then clearly $C \cdot C = 0$ on $\tilde{\Sigma}$.

So in the defining expression for $\mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d})$ we only need to sum the terms for $I \subset T$ with $\alpha_i \neq 0$ for all $i \in I$. A simple computation (or directly the formula [Ve5, Proposition 5.4]) yields

$$\begin{aligned}
\mathcal{E}_{\tilde{\Sigma}}(p^{-1}D, p^*\omega^{1/d}) &= L(L-1) + \sum_{i=1}^{2} \frac{L-1}{L^{\alpha_i}-1}(L-2+2(1+L^{\frac{\alpha_i+1}{2}})) \\
&= L(L-1)(1+\sum_{i=1}^{2}\frac{1}{L^{\alpha_i}-1}) + 2(L-1)\sum_{i=1}^{2}\frac{L^{\frac{\alpha_i+1}{2}}}{L^{\alpha_i}-1} \\
&= 0 + 2(L-1)L^{1/2}\frac{L^{\alpha_1+\frac{\alpha_2}{2}} - L^{\frac{\alpha_1}{2}} - L^{\frac{\alpha_2}{2}} + L^{\alpha_2+\frac{\alpha_1}{2}}}{(L^{\alpha_1}-1)(L^{\alpha_2}-1)} \\
&= 2(L-1)L^{1/2}\frac{(L^{\frac{\alpha_1+\alpha_2}{2}}-1)(L^{\frac{\alpha_1}{2}}+L^{\frac{\alpha_2}{2}})}{(L^{\alpha_1}-1)(L^{\alpha_2}-1)} = 0,
\end{aligned}$$

using twice that $\alpha_1 + \alpha_2 = 0$. □

We are now ready to prove our vanishing theorem.

2.3. Theorem. *Let X be a non–singular projective rational surface and $\omega^{1/d}$ a multi–valued differential form on X without logarithmic poles. Let $B = \cup_{i\in T} C_i$ be a connected normal crossings divisor on X satisfying $B \supset |\operatorname{div}\omega|$. If $\chi(X \setminus B) \leq 0$, then $PV \int_X \omega^{1/d}$ $(= \mathcal{E}_X(B, \omega^{1/d})) = 0$.*

Remark. The generalization with $B \supset |\operatorname{div}\omega|$ is not only needed for the applications in §3, but is already useful in the proof for $B = |\operatorname{div}\omega|$.

Proof. We first explain our strategy. We will construct maps $\Sigma \overset{g}{\leftarrow} \tilde{X} \overset{h}{\rightarrow} X$ as in the Structure Theorem 2.1. If the exceptional situation of Lemma 1.5(2) does not occur in any blowing–up of g or h, then this lemma implies

$$\mathcal{E}_X(B, \omega^{1/d}) = \mathcal{E}_{\tilde{X}}(h^{-1}B, \omega^{1/d}) = \mathcal{E}_{\tilde{X}}(h^{-1}B', \omega^{1/d}) = \mathcal{E}_\Sigma(g(h^{-1}B'), \omega^{1/d}),$$

where for simplicity we keep the notation $\omega^{1/d}$ on each surface (a rational differential form is a birational notion anyway). By Lemma 2.2 this last expression vanishes (the configuration on Σ turns out to be allowed). The crucial point is that we will show that indeed such an exceptional situation never occurs.

We will denote all curves appearing in the image of $h^{-1}B'$ in any intermediate surface by C_ℓ.

STEP 1. *The surface X does not contain any (-1)–curve disjoint from B.*

Indeed, suppose that A is such a (-1)–curve disjoint from B. Applying the adjunction formula for A on X would yield $-2 = \deg K_A = A \cdot A = -1$.

So we can really apply Theorem 2.1 and we obtain the extended configuration $B' \supset B$ on X and the maps $\Sigma \overset{g}{\leftarrow} \tilde{X} \overset{h}{\rightarrow} X$, using from now on all notations introduced in that theorem.

STEP 2. *The exceptional situation of Lemma 1.5(2) does not occur for* any *constituting blowing–up of $h : \tilde{X} \to X$.*

Suppose that there is such a blowing–up $b : X'' \to X'$ with centre $P_i \in C_i$, $\alpha_i = 0$, and two other curves C_{i_1} and C_{i_2} intersecting C_i outside P_i, satisfying $\alpha_{i_1} + \alpha_{i_2} = 0$ and $\{\alpha_{i_1}, \alpha_{i_2}\} \neq \{-1, 1\}$; see Figure 4. Denote by C the exceptional curve of b; recall that $\alpha = 1$.

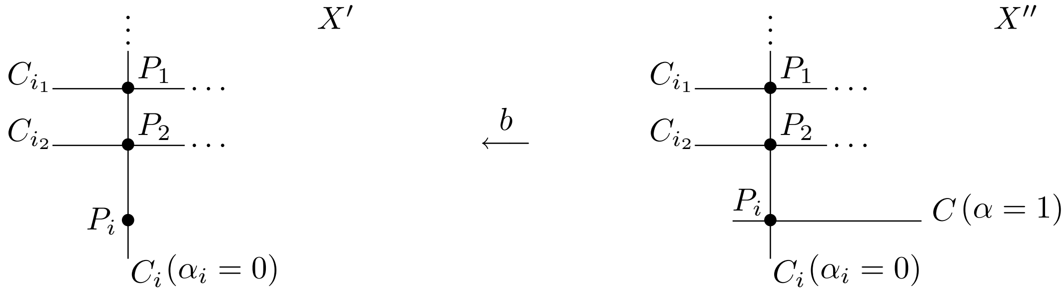


FIGURE 4

Since b is really needed to resolve the indeterminacies of φ (h is minimal), either C itself or some other curve, created after a chain of blowing–ups starting with centre a point of C, will project by g onto a section or bisection on Σ. Such a curve C' has $\alpha' \in \mathbb{Z}_{>0}$, and hence the configuration on Σ must be case (b). This also implies that C_i cannot project onto a section or bisection on Σ, so it is either blown down during g or becomes a fibre on Σ. Both possibilities yield that at some stage of g the curve C_i can intersect *at most two* other components of the total transform of B'.

So in order to reach that stage by blowing–downs starting from $\tilde{X}$, it is necessary that at least for one of the points P_1, P_2 or P_i (considered on $C_i \subset \tilde{X}$) the following holds: *$h^{-1}B' \setminus \{P_\ell\}$ has two connected components, and the one not containing C_i is a tree that gets completely contracted before reaching that stage of g.*

This cannot happen for P_i because, as explained above, either C itself or another curve in 'its tree' must project onto a section of Σ.

Consequently there is such a tree contracting to P_1 or P_2 during g, say to P_1. Now, since $\alpha_i = 0$, the component C_ℓ of $h^{-1}B$ through P_1 (in $\tilde{X}$) still has $\alpha_\ell = \alpha_{i_1}$. Analogously, at the stage of g just before the last blowing–down needed to contract the tree, the 'last' component C'_ℓ through P_1 still has $\alpha'_\ell = \alpha_{i_1} (\neq 1)$. But, on the other hand, since C'_ℓ now gets contracted by that blowing–down, we must have $\alpha'_\ell = 1$. This contradiction finishes step 2.

STEP 3. *The exceptional situation of Lemma 1.5(2) does not occur for any constituting blowing–up of* $g : \tilde{X} \to \Sigma$.

(To be precise, in case (c) for Σ we consider only constituting blowing–ups of $\tilde{X} \to \tilde{\Sigma}$, using the notation of (2.2).) Suppose that there is such a blowing–up $b : X'' \to X'$, where we use again the notations as in Figure 4, but this time 'during g'. We consider two cases.

(I) $P_i \in X'$ *does not belong to any other component of the total transform of* B'.

Here we consider two subcases.

(i) $C \subset h^{-1}B$ in $\tilde{X}$.

Since $C_i (\subset \tilde{X})$ must be blown down during h (all $\alpha_\ell \neq 0$ on X) and B has normal crossings, *at most two* components of the total transform of B can intersect C_i just before that blowing–down. As we explained above, since $\alpha_{i_1} \neq 1 \neq \alpha_{i_2}$, the components intersecting C_i in P_1 and P_2 cannot be blown down before C_i. So the tree that was created during g, starting with the blowing–up with centre P_i, must be contracted again by h. This contradicts the minimality of h.

(ii) $C (\subset \tilde{X})$ is a component of (the strict transform of) $B' \setminus B$.

Because $h^{-1}B$ is connected, each possible further blowing–up of g (after b) with centre in C must have P_i as centre. Consequently

$$\chi(X \setminus B') < \chi(X \setminus B) \leq 0$$

and Σ must be as in case (a) with *at least three fibres*. The strict transform in $\tilde{X}$ of the section $C_1 \subset \Sigma$ can intersect *at most two* other components of $h^{-1}B$. (Indeed, since $\alpha_i = 0$, the morphism h cannot be the identity. Hence C_1, being the only non–fibre in Σ, must be created by the last blowing–up of h.) So C_1 must intersect in $\tilde{X}$ at least one component C' coming from $B' \setminus B$. But C' gets contracted during g; so in order to have at least three intersections with C_1 in Σ, C' must intersect in $\tilde{X}$ *another* component of $h^{-1}B$. This contradicts Theorem 2.1(3).

(II) $P_i \in X'$ *also belongs to another component* C' *of the total transform of* B'.

Since, by Lemma 1.5, $(h^{-1}B, \omega^{1/d})$ is allowed on $\tilde{X}$, we then have necessarily $\alpha' = 1$. Moreover, we claim that C_i must already live on Σ, or on $\tilde{\Sigma}$ in case (c), where we use the notation of (2.2). Indeed, suppose that C_i is created by a blowing–up of g, or of $\tilde{X} \to \tilde{\Sigma}$ in case (c); at that stage C_i can intersect *at most two* other components. Thus C' should be created afterwards during g. But this situation is precisely the previous case (I), which we already contradicted.

We now investigate the cases (a), (b) and (c) for Σ in our (hypothetical) situation. Case (a) is clearly impossible. In case (b) C_i must be a section, and in case (c) C_i must be the bisection in Σ, see Figure 5. (In case (c) C_i could be a priori another curve in $\tilde{\Sigma}$, but then we would have two intersecting curves with $\alpha_\ell = 0$ on $\tilde{\Sigma}$, implying the same phenomenon on $\tilde{X}$.) Somewhat abusing notation in Figure 5, the curves C_{i_1}, C_{i_2} and C' in Σ could be different from those intersecting C_i in X', but their α–coefficients are also α_{i_1}, α_{i_2} and $\alpha'(= 1)$, respectively.

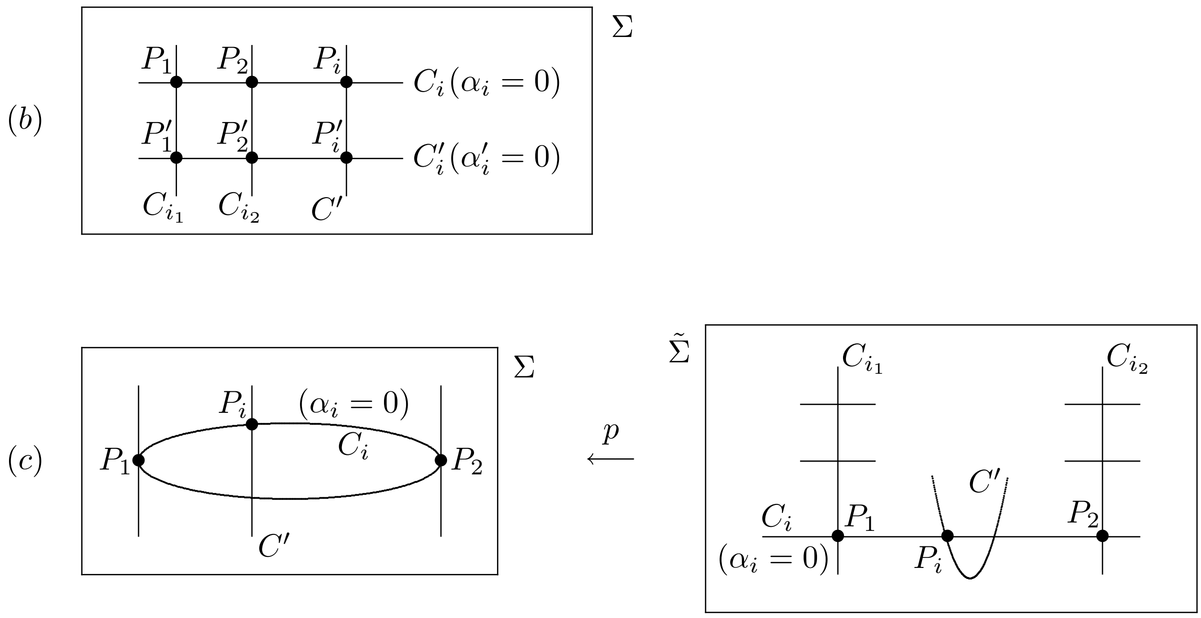


Figure 5

Case (b). On $\tilde{X}$ we have that C_i (being an exceptional curve of h) can intersect *at most two* other components of $h^{-1}B$. So all but at most two intersections of C_i in Σ with fibres C_ℓ of $\pi : \Sigma \to \mathbb{P}^1$ must 'split' during g, i.e. C_i must eventually get separated from those C_ℓ by a blowing–up of g with centre in C_i and exceptional curve coming from $B' \setminus B$. (The same is true for C'_i.)

In order to obtain such an exceptional curve C_j from $B' \setminus B$, intersecting C_i in P_1, we should have simultaneously $\alpha_j = 1$ and $\alpha_j = \alpha_{i_1} (\neq 1)$, by the same argument as for the conclusion of step 2. The same contradiction works for P_2, P'_1 and P'_2. Hence we must separate C_i and C'_i from C' with exceptional curves from $B' \setminus B$, intersecting C_i and C'_i in P_i and P'_i, respectively. But this contradicts the connectivity of $h^{-1}B$.

Case (c). Analogously we will have to separate C' from C_i, contradicting the connectivity of $h^{-1}B$.

Step 4. *$(g(h^{-1}B'),\omega^{1/d})$ is allowed on Σ and $\mathcal{E}_X(B,\omega^{1/d})=\mathcal{E}_\Sigma(g(h^{-1}B'),\omega^{1/d})=0$.*

By Lemma 1.5 we have $\mathcal{E}_X(B,\omega^{1/d})=\mathcal{E}_{\tilde X}(h^{-1}B,\omega^{1/d})$. Since also $h^{-1}B'$ is a normal crossings divisor and the curves C_ℓ in $B'\setminus B$ have $\alpha_\ell=1$, we have that also $(h^{-1}B',\omega^{1/d})$ is allowed on $\tilde X$ and $\mathcal{E}_{\tilde X}(h^{-1}B,\omega^{1/d})=\mathcal{E}_{\tilde X}(h^{-1}B',\omega^{1/d})$. It is easy to see that then necessarily $(g(h^{-1}B'),\omega^{1/d})$ is allowed on Σ, and thus $\mathcal{E}_{\tilde X}(h^{-1}B',\omega^{1/d})=\mathcal{E}_\Sigma(g(h^{-1}B'),\omega^{1/d})$, again by Lemma 1.5(2). This last expression equals zero by Lemma 2.2. $\square$

2.4. *Remark.* Theorem 2.3 and its proof are quite subtle. In [Ve7, 3.4] we constructed the following similar example. Let $X=\mathbb{P}^2$ and $\omega^{1/2}$ a multi–valued differential form with $|\operatorname{div}\omega|$ a (non–singular) conic B. So $\omega^{1/2}$ has no logarithmic poles on X and one easily computes that $PV\int_X\omega^{1/2}\neq 0$. Taking B' as the union of B and one tangent line to B, we can construct $\Sigma\xleftarrow{g}\tilde X\xrightarrow{h}X$, where h is a composition of three blowing–ups and g a composition of two blowing–downs, all outside of $X\setminus B'$, such that $g(h^{-1}B')\subset\Sigma$ is case (b) in Theorem 2.1 with exactly one fibre. Also on Σ we have that $\omega^{1/2}$ has no logarithmic poles, but now $PV\int_\Sigma\omega^{1/2}=0$. In this example we have that $\chi(X\setminus B')=0$; the only obstruction with the data of Theorem 2.3 is that here $\chi(X\setminus B)=1\,(>0)$. And in fact the 'change' in principal value integral is caused by g which consists precisely of the exceptional situation of Lemma 1.5(2).

2.5. Our vanishing theorem specializes to the level of Hodge polynomials or Euler characteristics. With the same notations as in Definition 1.4 we can introduce analogously the invariants

$$E_X(D,\omega^{1/d}):=\sum_{\substack{I\subset T\\ \forall i\in I:\alpha_i\neq 0}}H(C_I^\circ)\prod_{i\in I}\frac{uv-1}{(uv)^{\alpha_i}-1}+\sum_{\substack{i\in T\\ \alpha_i=0}}(-C_i\cdot C_i)\prod_{j\in T_i}\frac{uv-1}{(uv)^{\alpha_j}-1}$$

and

$$e_X(D,\omega^{1/d}):=\sum_{\substack{I\subset T\\ \forall i\in I:\alpha_i\neq 0}}\chi(C_I^\circ)\prod_{i\in I}\frac{1}{\alpha_i}+\sum_{\substack{i\in T\\ \alpha_i=0}}(-C_i\cdot C_i)\prod_{j\in T_i}\frac{1}{\alpha_j}.$$

Then with the same data as in Theorem 2.3 we obtain that $E_X(B,\omega^{1/d})=e_X(B,\omega^{1/d})=0$.

2.6. Let L be a p–adic field with valuation ring $\mathcal{O}$, maximal ideal P and residue field $\frac{\mathcal{O}}{P}\cong\mathbb{F}_q$. We choose an embedding of L into $\mathbb{C}$. When X and $\omega^{1/d}$ are defined over L, we can introduce analogously

$$E_X^L(D,\omega^{1/d}):=\sum_{\substack{I\subset T\\ \forall i\in I:\alpha_i\neq 0}}\operatorname{card}(C_I^\circ)_{\mathbb{F}_q}\prod_{i\in I}\frac{q-1}{q^{\alpha_i}-1}+\sum_{\substack{i\in T\\ \alpha_i=0}}(-C_i\cdot C_i)\prod_{j\in T_i}\frac{q-1}{q^{\alpha_j}-1},$$

where $\operatorname{card}(\cdot)_{\mathbb{F}_q}$ denotes the number of $\mathbb{F}_q$–rational points of the reduction mod P of C_I°. When $\omega^{1/d}$ has no logarithmic poles and if suitable conditions about good reduction mod

P are satisfied, a similar proof as for Denef's formula for the p–adic Igusa zeta function [De1] yields that $E_X^L(D,\omega^{1/d})$ is (up to a power of q) precisely the p–adic principal value integral $PV\int_{X(L)}|\omega^{1/d}|$.

When we assume that the field L is 'big enough', the same data as in Theorem 2.3 will again imply that this invariant vanishes. More precisely we want that all constructions in this proof of the theorem can be done 'over L', for instance that all centres of blowing–ups are L–rational points. This can always be achieved by taking a finite extension of a given p–adic field.

3. Cancelation of candidate poles for zeta functions

3.1. Fix a polynomial $f\in\mathbb{C}[X_1,\dots,X_{n+1}]\backslash\mathbb{C}$, or, more generally, a non–constant regular function $f:M\to\mathbb{A}^1$ from a non–singular $(n+1)$–dimensional variety M. Denef and Loeser [DL2] associated to f its *motivic zeta function* $\mathcal{Z}_{\text{mot}}(f;T)\in\mathcal{M}[[T]]$, where $\mathcal{M}$ is the localization of $K_0(Var)$ with respect to L. It describes the orders of all truncated arcs on M along the hypersurface $\{f=0\}$, and as in fact modeled on the classical p–adic Igusa zeta function. (See also [DL4] or [Ve6] for an introduction to this topic.) We just mention a formula for $Z_{\text{mot}}(f;T)$ in terms of an embedded resolution of $\{f=0\}$, implying in particular that this invariant belongs to the localization of $\mathcal{M}[T]$ with respect to L^a-T^b, where $a,b\in\mathbb{Z}_{>0}$.

3.2. Theorem [DL2, 2.2.1]. *Let $h:Y\to M$ be an embedded resolution of $\{f=0\}$. Let $E_j, j\in K$, be the irreducible components of $h^{-1}\{f=0\}$. Denote by N_j the multiplicity of E_j in* $\text{div}(f\circ h)$ *and by ν_j-1 the multiplicity of E_j in* $\text{div}(h^*dx)$*, where dx is a local generator of the sheaf of differential $(n+1)$–forms on M. For $J\subset K$ we put $E_J^\circ:=(\cap_{j\in J}E_j)\setminus(\cup_{\ell\notin J}E_\ell)$; so $Y=\coprod_{J\subset K}E_J^\circ$. Then*

$$\mathcal{Z}_{\text{mot}}(f;T)=L^{-(n+1)}\sum_{J\subset K}[E_J^\circ]\prod_{j\in J}\frac{(L-1)T^{N_j}}{L^{\nu_j}-T^{N_j}}.$$

3.3. This zeta function specializes to the *Hodge zeta function* $Z_{\text{Hod}}(f;T)$, replacing in the formula above all classes of varieties in $\mathcal{M}$ by their Hodge polynomial, and further to the 'classical' *topological zeta function*

$$z_{\text{top}}(f;s)=\sum_{J\subset K}\chi(E_J^\circ)\prod_{j\in J}\frac{1}{\nu_j+sN_j}\in\mathbb{Q}(s)$$

of [DL1]. We refer to [DL2] or [Ve6] for more details.

3.4. The famous *monodromy conjecture*, stated originally for the p–adic Igusa zeta function, can be formulated for $\mathcal{Z}_{\text{mot}}(f;T)$ as follows [DL2, 2.4]:

$Z_{\mathrm{mot}}(f;T)$ belongs to the localization of $\mathcal{M}$ with respect to those $L^a - T^b$, $a,b \in \mathbb{Z}_{>0}$, such that $e^{2\pi i a/b}$ is an eigenvalue of the local monodromy on the Milnor fibre of f at some point of $\{f=0\}$.

So if $L^{\nu/N}$ is a pole of $\mathcal{Z}_{\mathrm{mot}}(f;T)$, then $e^{2\pi i\nu/N}$ is expected to be an eigenvalue of the local monodromy. Note however that one has to be careful with the notion of pole here, the difficulty being that we do not know whether $\mathcal{M}$ is a domain. See e.g. [RV2] for a precise definition. For the Hodge and topological zeta function the notion of pole is clear.

The (p–adic version of) the conjecture was proved in dimension two ($n=1$) by Loeser [Loe1]; a simple proof in the motivic/Hodge/topological setting is in [Ro1]. It is still open in general, with partial results in dimension three [Ve2][RV1][ACLM1] in and other special cases [Loe2][ACLM2].

3.5. We keep using the notation of Theorem 3.2. Fix an exceptional component E_j of h which is mapped to a point by h. It induces the candidate pole L^{ν_j/N_j} for $\mathcal{Z}_{\mathrm{mot}}(f;T)$, respectively $(uv)^{\nu_j/N_j}$ for $Z_{\mathrm{Hod}}(f;T)$ and $-\nu_j/N_j$ for $z_{\mathrm{top}}(f;s)$.

In order for the monodromy conjecture to hold, looking at A'Campo's formula for the monodromy zeta function [A'C] one expects the following. Suppose we are in the generic case that $\nu_j/N_j \neq \nu_i/N_i$ for all $i \neq j$. If $\chi(E_j^\circ)=0$, maybe even if $(-1)^n\chi(E_j^\circ) \leq 0$, then 'in general' L^{ν_j/N_j} should not be a pole of $\mathcal{Z}_{\mathrm{mot}}(f;T)$.

Somewhat more precise, suppose only that $\nu_j/N_j \neq \nu_i/N_i$ for all $i \in S_j := \{i \in K \mid E_i \cap E_j \neq \emptyset\}$. Then, if $(-1)^n\chi(E_j^\circ) \leq 0$, one expects that 'in general' E_j does not contribute to the possible pole L^{ν_j/N_j}, which means that L^{ν_j/N_j} should not be a pole of

$$(*) \qquad \frac{1}{L^{n+1}} \sum_{j\in I\subset K} [E_I^\circ] \prod_{i\in I} \frac{(L-1)T^{N_i}}{L^{\nu_i} - T^{N_i}}.$$

We refer to e.g. [Ve2, §1] for a motivation. For $n=1$ this expectation is true and is part of the proof of the monodromy conjecture [Loe1][Ro1]. Now, L^{ν_j/N_j} not being a pole of $(*)$ can be reformulated as 'its residue is zero', i.e.

$$(**) \qquad \sum_{j\in I\subset K} [E_I^\circ] \prod_{i\in I\setminus\{j\}} \frac{L-1}{L^{\alpha_i}-1}$$

is zero, where $\alpha_i := \nu_i - (\nu_j/N_j)N_i$ for $i \in S_j$. Now $(**)$ is a motivic principal value integral on E_j. Indeed, let dx be a local generator of the sheaf of $(n+1)$–forms on M around the point $h(E_j)$. Then the Poincaré residue $\omega^{1/d}$ of $(f\circ h)^{-\nu_j/N_j}h^*(dx)$ on E_j is a multi–valued differential form on E_j with $\operatorname{div}\omega^{1/d} = \sum_{i\in S_j}(\alpha_i - 1)(E_j\cap E_i)$. This is easily verified with local coordinates, see also [Ja3]. We thus have that $(**)$ is (up to a non–zero constant) equal to $PV\int_{E_j}\omega^{1/d}$.

Note however that by assumption all $\alpha_i \neq 0$, so $\omega^{1/d}$ indeed has no logarithmic poles, but it is possible that some $\alpha_i = 1$. So in fact we land in a natural way in the more general framework of (1.3) with $D := \cup_{i\in S_j}(E_j\cap E_i) \supset |\operatorname{div}\omega|$, where the inclusion may be strict, and $(**)$ is (essentially) $\mathcal{E}_{E_j}(D,\omega^{1/d})$.

3.6. Let now $n = 2$. We may assume that h is constructed as a composition of blowing–ups with non–singular centre. Fix as above a projective exceptional surface E_j which is mapped to a point by h. The surface E_j was created during some blowing–up π of the resolution process h as a surface $E_j^{(0)}$, where either $E_j^{(0)} \cong \mathbb{P}^2$ or $E_j^{(0)}$ is a ruled surface, when the centre of π is a point or a curve, respectively. And then E_j is obtained from $E_j^{(0)}$ by a composition $\varphi : E_j \to E_j^{(0)}$ of (point) blowing–ups. Denote again $D := \cup_{i \in S_j}(E_j \cap E_i)$. We have that D is the inverse image by φ of the intersection of $E_j^{(0)}$ with the other components of the total inverse image of $\{f = 0\}$ at the stage of h when $E_j^{(0)}$ was just created. In particular D is connected if and only if this intersection on $E_j^{(0)}$ is connected. And it is thus always connected if $E_j^{(0)} \cong \mathbb{P}^2$.

3.7. By the considerations above Theorem 2.3 yields the following cancelation result for candidate poles of the motivic zeta function. For ease of reference we recall the notations.

Let M be a threedimensional non–singular variety and $f : M \to \mathbb{A}^1$ a non–constant regular function. Let $h : Y \to M$ be an embedded resolution of $\{f = 0\}$, constructed as a composition of blowing–ups. Denote by $E_j, j \in K$, the irreducible components of $h^{-1}\{f = 0\}$ and let N_j, ν_j and E_j° be as in (3.2). Suppose that E_j is mapped to a point by h and that $\nu_j/N_j \neq \nu_i/N_i$ for all $i \in S_j := \{i \in K \mid E_j \cap E_i \neq \emptyset\}$. Denote

$$R_{E_j} := \sum_{j \in I \subset K} [E_I^\circ] \prod_{i \in I \setminus \{j\}} \frac{L-1}{L^{\alpha_i} - 1},$$

'the contribution of E_j to the residue of L^{ν_j/N_j} for $\mathcal{Z}_{\text{mot}}(f;T)$'.

Theorem. *Let $\chi(E_j^\circ) \leq 0$.*

(1) If E_j is created by blowing up a point, then we have always $R_{E_j} = 0$.

(2) If E_j is created by blowing up a rational curve, and if $\cup_{i \in S_j}(E_j \cap E_i)$ is connected, then $R_{E_j} = 0$.

Recall that in case (2) this connectivity is equivalent to the connectivity of the analogous intersection configuration $D^{(0)}$ on the (rational) ruled surface $E_j^{(0)}$ (3.6). Note then that the exceptions in (2), i.e. those E_j with a non–connected intersection configuration, are very special! Indeed, non–connectivity of $D^{(0)}$ implies for instance that $D^{(0)}$ does not contain any fibre of the ruled surface $E_j^{(0)}$. In the embedded resolution process this is quite rare.

There is a recent vanishing result of Rodrigues [Ro2] in this exceptional case. When $\chi(E_j^\circ) = 0$, and assuming a minor extra condition, he classified all possible non–connected $D^{(0)}$ with non–singular irreducible components, and verified that then again $R_{E_j} = 0$.

3.8. The case where E_j is a rational surface is the most difficult one. There is no classification of the possible intersection configurations on E_j with $\chi(E_j^\circ) \leq 0$; instead we used our structure theorem 2.1. When E_j is created by blowing up a non–rational curve, we already obtained a classification of the possible configurations with $\chi(E_j^\circ) \leq 0$ in [Ve2] and [Ve3].

Proposition. *We use all notations of (3.7). Let* $\chi(E_j^\circ) \leq 0$*. If* E_j *is created by blowing up a non–rational curve, then we have always* $R_{E_j} = 0$*.*

Proof. Let as above E_j be created as the ruled surface $E_j^{(0)}$ while blowing up a curve of genus g during h. When $g \geq 2$, we classified the possible configurations on $E_j^{(0)}$ with $\chi(E_j^\circ) \leq 0$ in [Ve2, Proposition 5.13], and verified in [Ve2, Propositions 5.1 and 5.3] that $R_{E_j} = 0$ for them. (The calculation there is in the context of p–adic Igusa zeta functions, but is essentially the same in the motivic setting.)

When $g = 1$, the possible configurations $\mathcal{C}$ of curves on the ruled surface $E_j^{(0)}$ with $\chi(E_j \setminus \mathcal{C}) \leq 0$ were classified in [Ve3, Theorem 6.5]. Again by [Ve2, Propositions 5.1 and 5.3] we have that $R_{E_j} = 0$, except for one annoying case (where $R_{E_j} \neq 0$). More precisely, in this case the curves on the ruled surface consist of a number of disjoint elliptic curves, where either one of the curves is not a section, or there are at least three curves. Now recently Rodrigues showed in [Ro2] that in fact this configuration cannot occur in the context of exceptional surfaces in an embedded resolution. □

3.9. Theorem 3.7 and Proposition 3.8 specialize to the analogous results in the context of the Hodge and topological zeta functions of (3.3). They are also valid in the context of p–adic Igusa zeta functions (see e.g. [De1][Ve2]) if the p–adic field L is assumed 'big enough' as in (2.6), and if suitable conditions concerning good reduction mod P as in [De1] are satisfied. Alternatively, one can take a big enough number field F. Then our vanishing results will be true for the Igusa zeta functions over all except a finite number of completions L of F.

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